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Finite Rank Approximations to Integral Operators Which Satisfy Certain Total Positivity Conditions

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1. INTRODUCTION

1.1. *The Background*

In a paper published in 1969 Tikhomirov [21] determined the Kolmogorov widths (see Section 1.2), in the space $C([0, 1])$ of real continuous functions on an interval, of the sets

$$W_{\infty,r} = \{f \in C([0, 1]) : f^{(r-1)} \text{ is abs. continuous}, \|f^{(r)}\|_\infty \leqslant 1\},$$

$r = 1, 2, \ldots$. In a series of extremely interesting papers Micchelli and Pinkus generalized and extended Tikhomirov's results. The papers of Micchelli and Pinkus with which we are most concerned are [12, 13, 14, and 18]. Taylor's theorem provides a representation of each function $f \in W_{\infty,r}$ as

$$f(s) = k(s) + \int_0^1 \frac{(s-t)_+^{r-1}}{(r-1)!} f^{(r)}(t)\, dt,$$

where k is a polynomial of degree not greater than $r - 1$. The kernel

$$K(s, t) = \frac{(s-t)_+^{r-1}}{(r-1)!}$$

is totally positive [6]. Micchelli and Pinkus determined the Kolmogorov widths of sets determined by integral operators the kernels of which satisfy certain "total positivity" conditions. The integral operators which they considered in [12, 13, 14] act either from $L^\infty([0, 1])$ into $L^q([0, 1])$ $(1 \leqslant q \leqslant \infty)$ ([12, 14]) or from $L^p([0, 1])$ $(1 \leqslant p \leqslant \infty)$ into $L^1([0, 1])$ ([13, 14]). In [14] (and in passing, in [12]) they related the Kolmogorov widths to best approximations to the integral operators by operators of given finite rank.

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It is the purpose of the present account to show that all the results can be described systematically, and in a unified way, in terms of a restricted finite rank approximation to integral operators. This is achieved by considering a more general situation for which the principal "total positivity" condition takes a self-dual form (Section 2.1 Condition (C1)). In this situation the results for integral operators $T: L^p \to L^1$ are related precisely, by duality, to those for integral operators $T: L^\infty \to L^q$.

Tikhomirov [21], developing certain results of his first celebrated paper [20] on Kolmogorov widths, also determined all the Kolmogorov widths, in the space $\tilde{C}$ of real 2π-periodic continuous functions, of the sets

$$\tilde{W}_{\infty,r} = \{f \in \tilde{C}: f^{(r-1)} \text{ abs. continuous}, \|f^{(r)}\|_\infty \leqslant 1\},$$

$r = 1, 2,\ldots$. The present exposition is partly an outcome of examining the question whether any of the results developed by Micchelli and Pinkus in a non-periodic context has an analogue which generalises Tikhomirov's periodic result. The self-dual total positivity condition emerged from this examination and is precisely what is needed. However the theorem for the periodic situation which emerges is—relative to the non-periodic results—a rather special and restricted one concerning convolution operators. The only example we can give to which the result applies is that which is implicit in Tikhomirov's discussion—though we expect that other examples can be obtained theoretically by convolution with Cyclic Polya Frequency functions (see [6]).

This paper was first written before the author had seen the paper by Pinkus [18] which also considers the periodic situation. There is a non-trivial intersection between that paper and Section 3 of this one: the case $a = 0$ of Section 3 is contained in [18]. It is possible (or likely) that a unified treatment of the results of Section 3 and [18] could be developed. The present paper has been revised a little in the light of [18]: by developing one of the arguments of [18] and by giving effect to [18, Remark 3.1] it is shown that one of our original hypotheses (the conclusion of Theorem 3.1.1) is a consequence of the others. In revising the paper we have also added an abstract duality theorem (1.2.2).

The literature of n-widths is now extensive. Amongst those papers whose concerns are close to those of this paper there are those by Makovoz [11] and Ligun [9] which discuss n-widths in the periodic situation and use quite different methods, one by Micchelli and Pinkus [15] and a recent paper by Dyn [4] which develops some of the results of Micchelli and Pinkus in different directions; the latter two papers are explicitly concerned with finite rank approximations to integral operators.

In the remainder of the introduction, Section 1.2, the necessary definitions which relate to widths and finite rank approximation are formulated in an abstract context. The main non-periodic results are described in Section 2.1. A more precise description of their relation to the results of Micchelli and Pinkus is given at the end of Section 2.1. Section 2.3 is concerned with the "immediate" consequences of the total positivity conditions. Main Theorem 2.3.6 of this section can be described as a theorem of "variation diminishing type." Such theorems are discussed at length in [6]. The techniques and arguments used here are essentially straightforward extensions of those in [14]. They appear to be efficient; they apply with minor exceptions and with only notational variations to both the non-periodic and the periodic situations.

The heart of the non-periodic discussion in [21] is a variational problem. In [14] there are two. Section 2.4 is concerned with a common extension of those problems, and the development draws upon both [21, 14]. However, there are two cases which require some separate discussion. One relates to $L^\infty([0,1])$ and the other to $L^q([0,1])$ for $1 \leqslant q < \infty$. In the former case the discussion is of a slightly different nature to that in [21, 14] and the related paper [12]. At this point the general results relating to $L^\infty([0,1])$ fail to capture the entire information given by Tikhomirov's argument in the particular case considered by him.

1.2. *The Abstract Situation*

Let E and F be normed linear spaces. All the spaces with which we will be concerned are real, but the general definitions of this section apply equally to complex spaces. Let $\mathscr{L}(E,F)$ denote the set of bounded linear operators of E into F.

We begin with some definitions which are now standard. If $T \in \mathscr{L}(E,F)$ then, for each non-negative integer n,

$$a_n(T) = \inf\{\|T - T'\| : T' \in \mathscr{L}(E,F), \text{rank } T' \leqslant n\}.$$

In the terminology of Pietsch [16], $(a_n(T))_{n \geqslant 0}$ is the sequence of *approximation numbers* of T. If $T' \in \mathscr{L}(E,F)$, rank $T' \leqslant n$ and

$$\|T - T'\| = a_n(T)$$

we will say that "T' is extremal for $a_n(T)$".

If $x \in F$ and $L \subseteq F$ then the distance of x from L is

$$d(x, L) = d_F(x, L) = \inf\{\|x - y\| : y \in L\}.$$

If $X \subseteq F$ then the deviation of X from L is

$$\delta(X, L) = \delta_F(X, L) = \sup\{d(x, L) : x \in X\}.$$

The *Kolmogorov n-width* of X in F (originally defined in [7]) is

$$d_n(X, F) = \inf\{\delta(X, L) : L \text{ a subspace of } F, \dim L \leqslant n\}.$$

For any normed linear space E let E_1 denote the closed unit ball $\{x \in E : \|x\| \leqslant 1\}$ of E. If $T \in \mathscr{L}(E, F)$ the *Kolmogorov numbers* [16] of T are defined by

$$k_n(T) = d_n(T(E_1), F).$$

The sequences of approximation numbers and Kolmogorov numbers of an operator are examples of *s-number sequences* which were introduced and studied by Pietsch [16].

In order to discuss restricted finite rank approximation to certain integral operators we require extensions of these definitions. Let M_a be a subspace of F of finite dimension a and let N_b be a subspace of the dual E^* of E of finite dimension b. Then $N_b^\perp = \{x \in E : \langle x, f \rangle = 0 \text{ for all } f \in N_b\}$ is a closed linear subspace of codimension b in E. For $T \in \mathscr{L}(E, F)$ and $n \geqslant a$ define

$$a_n(T; M_a, N_b^\perp) = \inf\{\|T - T'\| : T' \in \mathscr{L}(E, F), \dim T'(N_b^\perp) \leqslant n, \quad M_a \subseteq T'(N_b^\perp)\},$$

and

$$k_n(T; M_a, N_b^\perp) = d_n(M_a + T(N_b^\perp \cap E_1), F).$$

Thus $a_n(T; \{0\}, E) = a_n(T)$ and $k_n(T; \{0\}, E) = k_n(T)$.

In the situation which will be considered the equality

$$a_n(T; M_a, N_b^\perp) = k_n(T; M_a, N_b^\perp)$$

will hold. The next lemma concerns relations which hold generally. Statements of the form "P is extremal for Q" will be made with their natural meaning.

Let $J : N_b^\perp \to E$ be the inclusion mapping, and let $\pi : F \to F/M_a$ be the quotient mapping. Then $\pi T J \in \mathscr{L}(N_b^\perp, F/M_a)$ is the composite

$$N_b^\perp \xrightarrow{J} E \xrightarrow{T} F \xrightarrow{\pi} F/M_a.$$

1.2.1. LEMMA. (i)

$$a_n(T; M_a, N_b^\perp) \geqslant a_{n-a}(\pi T J) \geqslant k_{n-a}(\pi T J) = k_n(T; M_a, N_b^\perp).$$

(ii) *If, for some $n \geqslant a$, $a_n(T; M_a, N_b^\perp) = k_n(T; M_a, N_b^\perp)$ and T' is extremal for $a_n(T; M_a, N_b^\perp)$ then $\pi T' J$ is extremal for $a_{n-a}(\pi T J)$ and the subspace $T'(N_b^\perp)$ is extremal for $k_n(T; M_a, N_b^\perp)$.*

The proof involves only straightforward calculations.

The final result of this section is a duality theorem.

1.2.2. THEOREM. *If* $\dim E \geqslant n+b$, $\dim F \geqslant n+b$ *and* $T \in \mathscr{L}(E,F)$ *is a compact linear operator then*

$$a_n(T; M_a, N_b^\perp) = a_{n-a+b}(T^*; N_b, M_a^\perp).$$

Proof. This result in the case in which $M_a = \{0\}$ and $N_b = \{0\}$ is attributed by Pietsch [16] to Hutton; all the elements of a proof are contained in [8, pp. 33–34].

The theorem will follow from the three inequalities

$$a_n(T; M_a, N_b^\perp) \underset{(1)}{\geqslant} a_{n-a+b}(T^*; N_b, M_a^\perp) \underset{(2)}{\geqslant} a_n(T^{**}; M_a, \hat{N}_b^\perp) \underset{(3)}{\geqslant} a_n(T; M_a, N_b^\perp).$$

It is convenient here to identify E and F with their canonical images in E^{**} and F^{**} and M_a (which is of finite dimension) with its second anihilator $(M_a^\perp)^\perp$ in F^{**}. The anihilators of N_b in E and E^{**} are denoted by $N_b^\perp$ and $\hat{N}_b^\perp$, respectively.

Consider the mappings

$$N_b^\perp \xrightarrow{J} E \xrightarrow{S} F \xrightarrow{\pi} F/M_a$$

and

$$M_a^\perp \xrightarrow{J'} F^* \xrightarrow{S^*} E^* \xrightarrow{\pi'} E^*/N_b,$$

where $S \in \mathscr{L}(E,F)$, $M_a \subseteq S(N_b^\perp)$ and $\dim S(N_b^\perp) \leqslant n$. There are isomorphisms $(F/M_a)^* \cong M_a^\perp$ and $(N_b^\perp)^* \cong E^*/N_b$. Therefore

$$\operatorname{rank}(\pi' S^* J') = \operatorname{rank}(\pi S J)^* = \operatorname{rank} \pi S J = \dim S(N_b^\perp) - a \leqslant n - a$$

and

$$\begin{aligned}\dim S^*(M_a^\perp) &= \operatorname{rank}(\pi' S^* J') + \dim S^*(M_a^\perp) \cap N_b \\ &\leqslant n - a + \dim S^*(M_a^\perp) \cap N_b.\end{aligned}$$

Now

$$\begin{aligned}\dim(\ker S^*) \cap M_a^\perp &= \dim M_a^\perp - \dim S^*(M_a^\perp) \\ &= \dim F^* - a - \dim S^*(M_a^\perp) \\ &\geqslant \dim F - n - \dim S^*(M_a^\perp) \cap N_b \\ &\geqslant b - \dim S^*(M_a^\perp) \cap N_b.\end{aligned}$$

It is now easily shown that for any $\varepsilon > 0$ there exists $S' \in \mathscr{L}(F^*, E^*)$ such that $\|S^* - S'\| < \varepsilon$, $N_b \subseteq S'(M_a^\perp)$ and $\dim S'(M_a^\perp) \leqslant n - a + b$. Then $\|T^* - S'\| \leqslant \|T^* - S^*\| + \varepsilon = \|T - S\| + \varepsilon$. This proves inequality (1). If

$\dim E \geqslant n+b$ then $\dim E^* \geqslant (n-a+b)+a$. Inequality (2) follows by applying (1) to $a_{n-a+b}(T^*; N_b, M_a^\perp)$ in place of $a_n(T; M_a, N_b^\perp)$.

Now suppose that $S \in \mathscr{L}(E^{**}, F^{**})$, $M_a \subseteq S(\hat{N}_b^\perp)$ and $\dim S(\hat{N}_b^\perp) \leqslant n$ and let $\varepsilon > 0$. Then $\|T - S|_E\| \leqslant \|T^{**} - S\|$ and $\dim S(N_b^\perp) \leqslant n - (a - \dim(S(N_b^\perp) \cap M_a))$. Also

$$\begin{aligned}\dim(\ker S \cap N_b^\perp) &= \dim N_b^\perp - \dim S(N_b^\perp)\\ &\geqslant \dim E - (n+b) + (a - \dim S(N_b^\perp) \cap M_a).\end{aligned}$$

Consequently there exists $S' \in \mathscr{L}(E, F^{**})$ such that $\|S|_E - S'\| < \varepsilon$ and $M_a \subseteq S'(N_b^\perp)$, $\dim S'(N_b^\perp) \leqslant n$. Thus $\|T - S'\| \leqslant \|T^{**} - S\| + \varepsilon$.

The operator T is compact so there exists a finite ε-net $\{y_k = Tx_k : k = 1,..., m\}$ for $T(E_1)$. Let $D = \text{sp}(S'(E) \cup \{y_1,..., y_m\})$. Then there exists $P : D \to F$ such that $\|D\| \leqslant 1 + \varepsilon$ and $Py = y$ for $y \in D \cap F$ ([8, 1.e.60]). Let $S'' = DS'$. If $x \in E_1$ then, for some j, $\|Tx - Tx_j\| \leqslant \varepsilon$ and

$$\begin{aligned}\|Tx - S''x\| &\leqslant \varepsilon + \|Tx_j - S''x\|\\ &= \varepsilon + \|DTx_j - DS'x\|\\ &\leqslant \varepsilon + (1+\varepsilon)\|Tx_j - S'x\|\\ &\leqslant \varepsilon + (1+\varepsilon)(\varepsilon + \|Tx - S'x\|)\\ &\leqslant 2\varepsilon + \varepsilon^2 + (1+\varepsilon)\|T - S'\|.\end{aligned}$$

Therefore $\|T - S''\| \leqslant 3\varepsilon + 2\varepsilon^2 + (1+\varepsilon)\|T^{**} - S\|$. Inequality (3) now follows.

2. Integral Operators between L^p Spaces

2.1. *Statement of Principal Results*

Henceforth linear spaces will be real and functions will be real-valued. The discussion in the non-periodic situation will be concerned with a continuous kernel $K \in C([0,1] \times [0,1])$ and functions $k_1,..., k_a$ and $g_1,..., g_b$ in $C([0,1])$. The spaces M_a and N_b will be $M_a = \text{sp}\{k_1,..., k_a\}$ and $N_b = \text{sp}\{g_1,..., g_b\}$. We will write

$$\mathscr{K} = (K; k_1,..., k_a; g_1,..., g_b)$$

and refer to $\mathscr{K}$ as "a system." It may happen that $a = 0$ or $b = 0$. We can indicate that $b = 0$, for example, by writing $\mathscr{K} = (K; k_1,..., k_a; \varnothing)$. To each kernel K and system $\mathscr{K} = (K; k_1,..., k_a; g_1,..., g_b)$ there are transposed kernel and system defined by

$$K'(s,t) = K(t,s), \qquad \mathscr{K}' = (K'; g_1,..., g_b; k_1,..., k_a).$$

If K and G are two kernels and f is an integrable function then we will use the notations $K * G$, $K * f$ and $f * K$, defined by

$$(K * G)(s, t) = \int_0^1 K(s, u)\, G(u, t)\, du,$$

$$(K * f)(s) = \int_0^1 K(s, t)\, f(t)\, dt,$$

$$f * K = K' * f.$$

The space $L^p([a, b])$ $(1 \leqslant p \leqslant \infty)$ will be denoted briefly by L^p and the norm of $f \in L^p$ by $\|f\|_p$. Note that in the notation of Section 1.2 the closed unit ball of, for example, L^1 is denoted $(L^1)_1$. If $1 \leqslant p \leqslant \infty$ then p' will denote the conjugate index given by $1/p + 1/p' = 1$.

If K is a continuous kernel then for each p, q with $1 \leqslant p, q \leqslant \infty$ there is an integral operator

$$T_K : L^p \longrightarrow L^q$$

defined by $T_K f = K * f$. We will denote by $|K|_{p,q}$ the mixed norm of K associated with this operator. Thus, if $1 < p \leqslant \infty$ and $1 \leqslant q < \infty$

$$|K|_{p,q} = \left(\int_0^1 \left(\int_0^1 |K(s, t)|^{p'}\, dt \right)^{q/p'} ds \right)^{1/q}.$$

If $p = 1$ or $q = \infty$ then suprema occur in place of integrals in the formula defining $|K|_{p,q}$. (This is not the standard use of the subscripts p, q in $|K|_{p,q}$, and, in particular, not the use of [14].) The operator norm of $T_K : L^p \to L^q$ will be denoted $\|T_K\|_{p,q}$. For reference we state an elementary fact as a proposition

2.1.1. PROPOSITION. *$\|T_K\|_{p,q} \leqslant |K|_{p,q}$, and equality holds in the case $q = \infty$.*

The principal result concerns approximations to K by finite rank kernels in the sense of $|\cdot|_{\infty,q}$ and restricted finite rank approximations to both T_K: $L^\infty \to L^q$ and $T_{K'}$: $L^{q'} \to L^1$. We make the following definitions:

$$a_n(\mathscr{K}; |\cdot|_{p,q}) = \inf\{|K - H|_{p,q} : \dim(H * (N_b^\perp \cap L^p)) \leqslant n, M_a \subseteq H * N_b^\perp\},$$

$$a_n(\mathscr{K}; \|\cdot\|_{p,q}) = a_n(T_K; M_a, N_b^\perp \cap L^p),$$

$$k_n(\mathscr{K}; p, q) = d_n(M_a + K * (N_b^\perp \cap (L^p)_1), L^q) = k_n(T_K; M_a, N_b^\perp \cap L^p),$$

where T_K denotes the integral operator in $\mathscr{L}(L^p, L^q)$ and $N_b^\perp$ now denotes the annihilator of N_b in L^1.

The following notation was used in [14]. Let

$$\Lambda_m = \{\tau = (\tau_1,..., \tau_m) \in \mathbb{R}^m: 0 < \tau_1 < \cdots < \tau_m < 1\}.$$

If $\tau \in \Lambda_m$ then it is sometimes (but in Section 2.4 not) convenient to put $\tau_0 = 0$ and $\tau_{m+1} = 1$. The greek letters ξ and τ, used *without subscripts*, will be reserved for points of some $\mathbb{R}^m$ with coordinates increasing, that is, points of the form $\tau = (\tau_1,..., \tau_m)$ with $\tau_1 \leqslant \tau_2 \leqslant \cdots \leqslant \tau_m$. If $\tau \in \Lambda_m^-$, the closure of Λ_m in $\mathbb{R}^m$, then h_τ will denote the step function defined by

$$\begin{aligned} h_\tau(t) &= (-1)^i \qquad \text{for} \quad \tau_i < t < \tau_{i+1} \text{ and } i = 0, 1,..., m, \\ &= 0 \qquad \text{for} \quad t = \tau_i \text{ and } i = 0, 1,..., m+1. \end{aligned}$$

It is convenient to observe at this point that

$$\begin{aligned} (K * h_\tau)(s) &= \int_0^1 K(s, t)\, h_\tau(t)\, dt \\ &= \sum_{i=1}^m 2(-1)^{i-1} \int_0^{\tau_i} K(s, t)\, dt + (-1)^m \int_0^1 K(s, t)\, dt. \end{aligned}$$

Note also that if $\tau \in \Lambda_m^-$ then $h_\tau = \pm h_{\tau'}$ for τ' in some Λ_σ, $\sigma \leqslant m$.

A major step in the development of the results which concern us was the introduction by Tikhomirov [21] of a certain variational problem. In the present situation we define, for $m \geqslant b$,

$$e_m(\mathscr{K}, q) = \inf\{\|k + K * h_\tau\|_q : k \in M_a, \tau \in \Lambda_m^-, h_\tau \in N_b^\perp\}.$$

The Hobby–Rice theorem (see [5, 17] and Remark 2.2.(1)) includes the assertion that $\{\tau \in \Lambda_m^-: \ h_\tau \in N_b^\perp\}$ is non-empty if $m \geqslant b$. Section 2.4 is concerned with the determination of a function $k_0 + K * h_{\tau^0}$ which is extremal for $e_m(\mathscr{K}, q)$.

The principal condition under which the results will hold is a total positivity condition on the system $\mathscr{K} = (K; k_1,..., k_a; g_1,..., g_b)$. An extension of the notations of [6] and [14] is required. If α, β, ρ and σ are non-negative integers such that $\alpha + \sigma = \beta + \rho$ then

$$\mathscr{K}\begin{pmatrix} i_1,..., i_\beta; \xi_1,..., \xi_\rho \\ j_1,..., j_\alpha; \tau_1,..., \tau_\sigma \end{pmatrix}$$

will denote the determinant

$$\begin{vmatrix} 0 & \cdots & 0 & g_{i_1}(\tau_1) & \cdots & g_{i_1}(\tau_\sigma) \\ \vdots & & \vdots & \vdots & & b \\ 0 & \cdots & 0 & g_{i_\beta}(\tau_1) & \cdots & g_{i_\beta}(\tau_\sigma) \\ k_{j_1}(\xi_1) & \cdots & k_{j_\alpha}(\xi_1) & K(\xi_1,\tau_1) & \cdots & K(\xi_1,\tau_\sigma) \\ \vdots & & \vdots & \vdots & & \vdots \\ k_{j_1}(\xi_\rho) & \cdots & k_{j_\alpha}(\xi_\rho) & K(\xi_\rho,\tau_1) & \cdots & K(\xi_\rho,\tau_\sigma) \end{vmatrix}.$$

We can allow, for example, $\beta = 0$ and write

$$\mathscr{K}\begin{pmatrix} \varnothing; & \xi_1,\dots,\xi_\rho \\ j_1,\dots,j_\alpha; & \tau_1,\dots,\tau_\sigma \end{pmatrix}.$$

In the notation of [6]

$$\mathscr{K}\begin{pmatrix} \varnothing; & \xi_1,\dots,\xi_\rho \\ \varnothing; & \tau_1,\dots,\tau_\rho \end{pmatrix} = K\begin{pmatrix} \xi_1,\dots,\xi_\rho \\ \tau_1,\dots,\tau_\rho \end{pmatrix}.$$

The main condition on systems $\mathscr{K}$ which enter into the discussion are conditions ($C1$) (a "total positivity" condition), ($C2$) (referred to in [14] as a "non-degeneracy" condition) and ($C3$). However, when $b \neq 0$ the full force of ($C2$) is invoked at only one point in the argument. Conditions ($C4$), ($C5$) and ($C6$) are formulated in order to indicate precisely what the proofs require.

In the statements of the conditions ρ and σ are integers such that $\rho - a = \sigma - b \geqslant 0$.

Condition ($C1(a+\sigma)$). If $\xi \in \Lambda_\rho$ and $\tau \in \Lambda_\sigma$ then

$$\mathscr{K}\begin{pmatrix} 1,\dots,b; & \xi_1,\dots,\xi_\rho \\ 1,\dots,a; & \tau_1,\dots,\tau_\sigma \end{pmatrix} \geqslant 0.$$

Condition (*Strict* $C1(a+\sigma)$).
If $0 \leqslant \xi_1 < \cdots < \xi_\rho \leqslant 1$ and $0 \leqslant \tau_1 < \cdots < \tau_\sigma \leqslant 1$ then

$$\mathscr{K}\begin{pmatrix} 1,\dots,b; & \xi_1,\dots,\xi_\rho \\ 1,\dots,a; & \tau_1,\dots,\tau_\sigma \end{pmatrix} > 0.$$

Condition ($C2(a+\sigma)$). If $\tau \in \Lambda_\sigma$ then the functions $k_1,\dots,k_a$, $K(\cdot,\tau_1),\dots,$ $K(\cdot,\tau_\sigma)$ are linearly independent.

Condition ($C3$). If $\tau \in \Lambda_b$ then

$$\mathscr{K}\begin{pmatrix} 1,\dots,b; & \varnothing \\ \varnothing; & \tau_1,\dots,\tau_b \end{pmatrix} = \det(g_i(\tau_j)) \neq 0.$$

This means that the functions $g_1,\ldots, g_b$ satisfy the Haar condition on the *open* interval (0, 1), and therefore form a weak Chebyshev system. If $b=0$ Condition ($C3$) is to be interpreted as vacuous.

The next condition is weaker than ($C2$) if $b \neq 0$ and at all but one point of the argument is adequate.

Condition ($C4(a+\sigma)$). If $\tau \in \Lambda_\sigma$ and $\alpha_1,\ldots,\alpha_a, \beta_1,\ldots,\beta_\sigma$ are coefficients, not all zero, such that

$$\sum_{j=1}^{\sigma} \beta_j g(\tau_j)=0 \qquad \text{for all} \quad g \in N_b$$

then

$$\sum_{j=1}^{a} \alpha_j k_j + \sum_{j=1}^{\sigma} \beta_j K(\cdot, \tau_j) \neq 0.$$

Conditions ($C3$) and ($C4(a+\sigma)$) are related to the next condition.

Condition ($C5(a+\sigma)$). If $\tau \in \Lambda_\sigma$ then there exists $\xi \in \Lambda_\rho$ such that

$$\mathscr{K}\begin{pmatrix} 1,\ldots, b;\ \xi_1,\ldots,\xi_\rho \\ 1,\ldots, a;\ \tau_1,\ldots,\tau_\sigma \end{pmatrix} \neq 0.$$

Finally, weaker than ($C5$) is

Condition ($C6(a+\sigma)$). There exist $\tau \in \Lambda_\sigma$ and $\xi \in \Lambda_\rho$ such that

$$\mathscr{K}\begin{pmatrix} 1,\ldots, b;\ \xi_1,\ldots,\xi_\rho \\ 1,\ldots, a;\ \tau_1,\ldots,\tau_\sigma \end{pmatrix} \neq 0.$$

Transposed conditions. If the transposed system $\mathscr{K}'$ satisfies ($C1(a+\sigma)$) we will say that $\mathscr{K}$ satisfies ($C1'(a+\sigma)$). Similarly with the other conditions.

Extended conditions. In [14] it is required that ($C2$) or ($C2'$) should extend to one of the end points of the interval [0, 1]. The extended condition is not essential to the argument. However, when it is satisfied additional information can be obtained. So we formulate

Condition (Ext $C4(a+\sigma)$). As ($C4(a+\sigma)$) but with "$\tau \in \Lambda_\sigma$" replaced by "$0<\tau_1<\cdots<\tau_\sigma \leqslant 1$." (Systems for which the condition ($C4(a+\sigma)$) extends to the left-hand end point of [0, 1] can be accommodated by a change of variable.)

Some of the relations between these conditions will be stated as a proposition.

2.1.2. PROPOSITION. (i) $(C2(a+\sigma)) \Rightarrow (C4(a+\sigma))$.

(ii) $(C3)$ *and* $(C4(a+\sigma)) \Rightarrow (C5(a+\sigma)) \Rightarrow (C6(a+\sigma))$; $(C5(a+b)) \Rightarrow (C3)$; $(C5(a+\sigma)) \Rightarrow (C4(a+\sigma))$.

(iii) (*Strict* $C1(a+\sigma)) \Rightarrow (C1(a+\sigma))$ *and* $(C5(a+\sigma))$.

The proof of the first part of (ii) is elementary but perhaps not trivial linear algebra.

Blanket conditions. The preceding conditions are formulated in such a way that they will apply with minimum modification to the periodic situation discussed in Section 3. For the non-periodic situation it is convenient to formulate

Condition $(C1)$. $\mathscr{K}$ satisfies $(C1(a+\sigma))$ for all $\sigma \geqslant \max\{b, 1\}$. We can use $(C2)$, $(C4)$, etc., in a similar way.

Finally we must note that if $a = b = 0$ then $(C1)$ is the condition that K be *totally positive*, and (Strict $C1$) the condition that K be *strictly totally positive*.

2.1.3. EXAMPLES. (1) The system

$$\mathscr{K} = \left(\frac{(s-t)_+^{r-1}}{(r-1)!}; 1,..., s^{r-1}; \varnothing\right)$$

satisfies conditions $(C1)$, $(C3)$ and (Ext $C2'$), it satisfies condition $(C2)$ extended to the left-hand end point of $[0, 1]$. Tikhomirov's paper [21] is primarily concerned with $k_n(\mathscr{K}; \infty, \infty)$ for this system.

(2) It can be shown that with a suitable choice of signs depending upon r, the system

$$\mathscr{K} = \left(\pm \frac{(s-t)_+^{r-1}}{(r-1)!}; 1,..., s^{r-1}; \pm 1,..., \pm t^{r-1}\right)$$

satisfies conditions $(C1)$, $(C3)$, $(C3')$, (Ext $C2'$) and condition $(C2)$ extended to the left-hand end point of $[0, 1]$.

(3) The kernel

$$\begin{aligned} K(s,t) &= s(1-t), \qquad 0 \leqslant s \leqslant t, \\ &= t(1-s), \qquad t \leqslant s \leqslant 1, \end{aligned}$$

is totally positive. In this case the system $\mathscr{K} = (K; \varnothing; \varnothing)$ satisfies the principal conditions of Theorems 2.1.4 and 2.1.5, but it does not satisfy the extended conditions and evades the requirements of [14].

(4) Further examples are provided by Green functions and eigenfunctions of certain differential operators (see [6, Chap. 6]).

It is now possible to summarise the results in two composite theorems.

2.1.4. THEOREM. *Suppose that the system $\mathscr{K}$ satisfies conditions (C1), (C3), (C4) and (C4′). Then for each integer $m \geqslant b$ there exists $k_0 \in M_a$, an integer σ with $b \leqslant \sigma \leqslant m$, and $\tau^0 \in \Lambda_\sigma$ with $h_{\tau^0} \in N_b^{\perp}$, such that*

(i) *$P_0 = k_0 + K * h_{\tau^0}$ is extremal for $e_m(\mathscr{K}, q)$,*

(ii) *P_0 has precisely $\rho = a - b + \sigma$ zeros in $(0, 1)$ at the points of some $\xi^0 = (\xi_1^0, ..., \xi_\rho^0) \in \Lambda_\rho$ and*

$$\mathscr{K}\begin{pmatrix} 1,..., b; & \xi_1^0, ..., \xi_\rho^0 \\ 1,..., a; & \tau_1^0, ..., \tau_\sigma^0 \end{pmatrix} > 0,$$

(iii) *in the case that $q = \infty$ there are $\rho + 1$ points of $[0, 1]$ at which P_0 attains its bound $\|P_0\|_\infty$ with alternating sign.*

If $1 \leqslant q < \infty$ then $e_m(\mathscr{K}, q) < e_{m-2}(\mathscr{K}, q)$ for $m \geqslant b + 2$. If $1 \leqslant q < \infty$ and $\mathscr{K}$ also satisfies (Ext C4) then $e_m(\mathscr{K}, q) < e_{m-1}(\mathscr{K}, q)$ for $m \geqslant b + 1$.

If $\mathscr{K}$ also satisfies condition (Strict C1) then $e_m(\mathscr{K}, \infty) < e_{m-1}(\mathscr{K}, \infty)$ for $m \geqslant b + 1$.

This theorem is essentially a summary of Section 2.4. It is given by Theorems 2.4.2, 2.4.3, 2.4.5 (with 2.3.6) and Lemma 2.3.7.

2.1.5. THEOREM. *Suppose that the system $\mathscr{K}$ satisfies conditions (C1), (C3), (C4) and (C2′). Suppose that $1 \leqslant q \leqslant \infty$ and $n \geqslant a$. Then*

$$a_n(\mathscr{K}; |\cdot|_{\infty,q}) = a_n(\mathscr{K}; \|\cdot\|_{\infty,q}) = k_n(\mathscr{K}; \infty, q) = e_{n-a+b}(\mathscr{K}, q).$$

Let P_0, $\tau^0 \in \Lambda_\sigma$ $(b \leqslant \sigma \leqslant n - a + b)$, $\xi^0 \in \Lambda_\rho$ $(\rho = a - b + \sigma)$ be as in Theorem 2.1.4 (applied to $m = n - a + b$). Then the kernel H_0 defined by

$$H_0(s, t) = K(s, t) - \frac{\mathscr{K}\begin{pmatrix} 1,..., b; & \xi_1^0, ..., \xi_\rho^0, s \\ 1,..., a; & \tau_1^0, ..., \tau_\sigma^0, t \end{pmatrix}}{\mathscr{K}\begin{pmatrix} 1,..., b; & \xi_1^0, ..., \xi_\rho^0 \\ 1,..., a; & \tau_1^0, ..., \tau_\sigma^0 \end{pmatrix}}$$

is extremal for $a_n(\mathscr{K}; |\cdot|_{\infty,q})$, and the integral operator $T_{H_0} \in \mathscr{L}(L^\infty, L^q)$ is extremal for $a_n(\mathscr{K}; \|\cdot\|_{\infty,q})$.

Let L_0 be the subspace

$$L_0 = \left\{ \sum_{j=1}^{a} \alpha_j k_j + \sum_{j=1}^{\sigma} \beta_j K(\cdot, \tau_j^0) : \sum_{j=1}^{\sigma} \beta_j g(\tau_j^0) = 0 \text{ for } g \in N_b \right\}$$

of $C([0, 1])$. Then L_0 is of dimension ρ and interpolates at the points of $\xi^0 = (\xi_1^0, ..., \xi_\rho^0)$; let $T_0 : C([0, 1]) \to L_0$ be the corresponding interpolation operator. Then $T_{H_0} | N_b^\perp = T_0(T_K | N_b^\perp)$ and this operator is extremal for $a_n(T_K | N_b^\perp; M_a, N_b^\perp)$; the subspace L_0 is extremal for $k_n(\mathscr{K}; \infty, q)$.

Suppose that the system $\mathscr{K}$ also satisfies conditions (C3′) and (C2). Then

$$a_{n-a+b}(\mathscr{K}'; \|\cdot\|_{q',1}) = k_{n-a+b}(\mathscr{K}'; q', 1) = e_{n-a+b}(\mathscr{K}, q).$$

Let L_0' be the subspace

$$L_0' = \left\{ \sum_{i=1}^{b} \alpha_i g_i + \sum_{i=1}^{\rho} \beta_i K(\xi_i^0, \cdot) : \sum_{i=1}^{\rho} \beta_i k(\xi_i^0) = 0 \text{ for } k \in M_a \right\}$$

of $C([0, 1])$. Then L_0' is of dimension σ and interpolates at the points of $\tau^0 = (\tau_1^0, ..., \tau_\sigma^0)$; let $T_0' : C([0, 1]) \to L_0'$ be the corresponding interpolation operator. Then $T_{H_0'}$ is extremal for $a_{n-a+b}(\mathscr{K}'; \|\cdot\|_{q',1})$. The operator $T_{H_0'} | M_a^\perp = T_0'(T_K | M_a^\perp)$ and is extremal for $a_{n-a+b}(T_K | M_a^\perp; N_b, M_a^\perp)$. The subspace L_0' is extremal for $k_{n-a+b}(\mathscr{K}'; q', 1)$.

Outline of Proof. The first assertion of the theorem follows from a succession of inequalities:

$$\begin{aligned} a_n(\mathscr{K}; |\cdot|_{\infty,q}) &\geqslant a_n(\mathscr{K}; \|\cdot\|_{\infty,q}) & (1)\\ &\geqslant k_n(\mathscr{K}; \infty, q) & (2)\\ &\geqslant e_{n-a+b}(\mathscr{K}, q) & (3)\\ &= \|P_0\|_q & (4)\\ &= |K - H_0|_{\infty,q} & (5)\\ &\geqslant a_\rho(\mathscr{K}; |\cdot|_{\infty,q}) & (6)\\ &\geqslant a_n(\mathscr{K}; |\cdot|_{\infty,q}). & (7) \end{aligned}$$

Inequality (1) is by 2.1.1. Inequality (2) is by 1.2.1. Inequality (3) is Theorem 2.2.1. Inequality (4) is by choice of P_0 (Theorem 2.1.4). H_0 is well-defined by 2.1.4(ii). Inequality (5) is by 2.3.8(iv). Inequality (6) is by 2.3.8(i) and (ii) which assert that the kernel H_0 satisfies the defining conditions of $a_\rho(\mathscr{K}; |\cdot|_{\infty,q})$. Inequality (7) is because $\rho \leqslant n$. It follows that all the inequalities are in fact equalities. The properties of H_0, T_{H_0}, L_0 and

$T_0(T_{H_0} \mid N_b^\perp)$ which are asserted now follow simply from 2.1.1, 1.2.1 and 2.3.8.

Now suppose that the system $\mathscr{K}$ also satisfies conditions $(C3')$ and $(C2)$. There is a succession of inequalities:

$$\begin{aligned} a_{n-a+b}(\mathscr{K}'; \|\cdot\|_{q',1}) &\geqslant k_{n-a+b}(\mathscr{K}'; q', 1) && (8)\\ &\geqslant e_{n-a+b}(\mathscr{K}, q) && (9)\\ &= |K - H_0|_{\infty,q} && (10)\\ &\geqslant \| T_K - T_{H_0}\|_{\infty,q} && (11)\\ &= \| T_{K'} - T_{H_0'}\|_{q',1} && (12)\\ &\geqslant a_\sigma(\mathscr{K}'; \|\cdot\|_{q',1}) && (13)\\ &\geqslant a_{n-a+b}(\mathscr{K}'; \|\cdot\|_{q',1}). && (14) \end{aligned}$$

Inequality (8) is by 2.1.1. Inequality (9) is Theorem 2.2.3. Equality (10) is (4) and (5) above. Inequality (11) is by 2.1.1. Equality (12) is an elementary duality result. Inequality (13) is by Theorem 2.3.8 applied to the transposed system $\mathscr{K}'$ (this step requires $(C3')$ and $(C2)$). Inequality (14) is because $\sigma \leqslant n-a+b$. It follows that there is equality throughout. The extremal properties of H_0', $T_{H_0'}$, L_0', etc., follow simply.

We remark that the equality

$$a_{n-a+b}(\mathscr{K}'; \|\cdot\|_{q',1}) = a_n(\mathscr{K}, \|\cdot\|_{\infty,q})$$

is essentially a special case of Theorem 1.2.2 (the case $q=1$ requires an appeal to symmetry). Thus the first part of the theorem, together with Theorem 1.2.2, allows us to bypass inequalities (8) and (9) and so obtain most of the theorem without the use of Theorem 2.2.3.

The relation of these results to those of Micchelli and Pinkus [12, 13, 14] will be described using the terminology of this paper. That part of [12] which relates to integral operators can be described as being concerned with $k_n(\mathscr{K}; \infty, \infty)$ and $a_n(\mathscr{K}, \infty, \infty)$ in situations in which $b=0$. The paper [13] is primarily concerned with $k_n(\mathscr{K}; 1, 1)$ in situations in which $b=0$ (or, one can say, with $k_n(\mathscr{K}'; 1, 1)$ when $a=0$). The introduction of the subspaces N_b unifies these results. The case $q=\infty$ of Theorem 2.1.4 contains both [12, Theorem 7.1] and [13, Theorem 2.1], but the proof is of a different nature. The paper [14] is concerned with $k_n(\mathscr{K}, \infty, q)$ $(b=0)$ and $k_n(\mathscr{K}'; q', 1)$ $(a=0)$ but in that paper these are related to $a_n(\mathscr{K}; \infty, q)$ and $a_n(\mathscr{K}'; q', 1)$ only when $a=b=0$. Theorem 2.1.5 (together with Theorem 2.1.4) contains a major part of [12, Theorem 7.2; 13, Theorem 2.2] and essentially all of the results of [14] apart from those which are concerned with Gel'fand widths of sets. (The results for Gel'fand widths can also be extended and incorporated.)

Finally we remark that Tikhomirov's conclusions [21] imply that, for the system

$$\mathscr{X} = \left(\frac{(s-t)_+^{r-1}}{(r-1)!}; 1, \ldots, s^{r-1}; \varnothing \right),$$

$e_m(\mathscr{X}, \infty) < e_{m-1}(\mathscr{X}, \infty)$ for $m > r$. This conclusion escapes Theorem 2.1.4.

2.2. *Applications of the Borsuk–Ulam Theorem*

The Borsuk–Ulam antipodal mapping theorem states that if $\varphi: S^n \to \mathbb{R}^n$ is a continuous and odd mapping of the euclidean n-sphere S^n into the euclidean space $\mathbb{R}^n$ then $0 \in \varphi(S^n)$. The theorem was first used in the exact determination of Kolmogorov widths by Tikhomirov [20]. In fact the determination of Kolmogorov widths and the Borsuk–Ulam theorem are inseparable (see [2]). It is convenient to use here a set-valued version of the theorem due to Day [3] (it can also be proved in an elegant way using the methods of Browder [1, cf. Theorem 4]):

Let φ be an upper semi-continuous non-empty compact convex set-valued mapping of S^n into $\mathbb{R}^n$ such that $\varphi(-x) = -\varphi(x)$ for all $x \in S^n$. Then there exists $x \in S^n$ such that $0 \in \varphi(x)$.

In the two theorems of this section $\mathscr{X}$ is a system as in Section 2.1, but no conditions are imposed on it.

2.2.1. THEOREM. *For each q, $1 \leqslant q \leqslant \infty$, and each integer $n \geqslant a$*

$$k_n(\mathscr{X}; \infty, q) \geqslant e_{n-a+b}(\mathscr{X}, q).$$

Proof. Let $\sigma = n - a + b$. First we introduce a mapping $\psi: S^\sigma \to L^\infty$ which is by Pinkus ([17], see also [14]) out of Hobby–Rice [5]. If

$$z = (z_1, \ldots, z_{\sigma+1}) \in S^\sigma = \{(z_1, \ldots, z_{\sigma+1}) \in \mathbb{R}^{\sigma+1}: z_1^2 + \cdots + z_{\sigma+1}^2 = 1\}$$

let $t_0(z) = 0$ and $t_i(z) = \sum_{j=1}^{i} z_j^2$ for $i = 1, \ldots, \sigma + 1$. Define $\psi(z) \in L^\infty$ by

$$\psi(z)(t) = \operatorname{sgn} z_j \qquad \text{for} \quad t \in (t_{j-1}(z), t_j(z)) \text{ and } j = 1, \ldots, \sigma + 1.$$

Then ψ is an odd mapping of S^σ into the unit ball $(L^\infty)_1$ of L^∞. It is continuous with respect to the L^1-norm on L^∞. Furthermore, for each $z \in S^\sigma$, $\psi(z) = \pm h_\tau$ for some $\tau \in \Lambda_\sigma^-$.

Let L be any $(n-a)$-dimensional subspace of the quotient space L^q/M_a and let P be the set-valued metric projection of L^q/M_a into L, that is, $P(x) = \{y \in L: \|x - y\| = d(x, L)\}$ for each $x \in L^q/M_a$. Then P is an upper semi-continuous non-empty compact convex set-valued mapping and it is odd. Let φ_1 be the set-valued mapping which is the composite

$$S^\sigma \xrightarrow{\psi} L^\infty \xrightarrow{T_K} L^q \xrightarrow{\pi} L^q/M_a \xrightarrow{P} L.$$

The composite $T_K\psi$ is continuous and so φ_1 is upper semi-continuous. Now define a set-valued mapping φ of S^σ into $\mathbb{R}^b \times L \cong \mathbb{R}^\sigma$ by

$$\varphi(z) = (\langle g_1, \psi(z)\rangle, \ldots, \langle g_b, \psi(z)\rangle) \times \varphi_1(z).$$

The mapping φ satisfies the conditions of the set-valued version of the Borsuk–Ulam theorem. Thus there exists $z \in S^\sigma$ such that $\psi(z) \in N_b^\perp$ and $0 \in \varphi_1(z)$. This latter condition means that

$$d(\pi T_K \psi(z), L) = \|\pi T_K \psi(z)\|.$$

Therefore for some $k_0 \in M_a$

$$d(k_0 + T_K \psi(z), \pi^{-1}(L)) = \|k_0 + T_K \psi(z)\|.$$

This proves that

$$\delta(M_a + T_K(N_b^\perp \cap (L^\infty)_1), \pi^{-1}(L)) \geqslant \|k_0 + T_K \psi(z)\|.$$

But for some $\tau \in \Lambda_\sigma^-$, $h_\tau = \pm\psi(z) \in N_b^\perp$, so

$$\|k_0 + T_K \psi(z)\| \geqslant e_\sigma(\mathscr{K}; q).$$

The theorem now follows.

2.2.2. *Remarks.* (1) The argument above essentially contains Pinkus's proof of the Hobby–Rice theorem. In the case $n = a$ it yields the conclusion that $h_\tau \in N_b^\perp$ for some $\tau \in \Lambda_b^-$. This is the consequence of the Hobby–Rice theorem to which we appealed when defining $e_m(\mathscr{K}, q)$ in Section 2.1.

(2) The set-valued version of the Borsuk–Ulam theorem can be avoided. The image $(\pi T_K \psi)(S^\sigma)$ is separable. Introduce an equivalent strictly convex norm $|||\cdot|||$ on the separable space $E = \operatorname{span}(L \cup (\pi T_K \psi)(S^\sigma)) \subseteq L^q/M_a$. The metric projection P_ε: $E \to L$ with respect to the norm $\|\cdot\| + \varepsilon|||\cdot|||$ is point-valued. Apply the Borsuk–Ulam theorem itself and select a cluster point as $\varepsilon \to 0$.

2.2.3. THEOREM. *For each q, $1 \leqslant q \leqslant \infty$ and each integer $n \geqslant b$*

$$k_n(\mathscr{K}'; q', 1) \geqslant e_n(\mathscr{K}, q).$$

Proof. Let $E = L^{q'}$ and $F = L^1$. Denote the integral operator

$$E = L^{q'} \xrightarrow{T_{K'}} L^1 = F$$

by $T \in \mathscr{L}(E, F)$. The conjugate mapping $T^* \in \mathscr{L}(F^*, E^*)$ is the composite

$$F^* \xrightarrow{\cong} L^\infty \xrightarrow{T_K} L^q \xrightarrow{J} E^* = (L^{q'})^*,$$

where J is the canonical embedding. First note the

PROPOSITION. *If $\psi \in L^q$ then $\|J\psi \mid (M_a^\perp \cap E)\| = d_{L^q}(\psi, M_a)$.*

There are two cases to consider. If $1 < q \leqslant \infty$ then J is an isometric isomorphism, $(M_a^\perp \cap E)^\perp = JM_a$ because $\dim M_a < \infty$, and

$$\|J\psi \mid (M_a^\perp \cap E)\| = d(J\psi, (M_a^\perp \cap E)^\perp) = d(J\psi, JM_a) = d(\psi, M_a).$$

If $q = 1$ then $E \cong (L^q)^*$ and the proposition follows from the fact that $(L^q/M_a)^* \cong (M_a^\perp \cap E)$.

Now let L be any n-dimensional subspace of F such that $N_b \subseteq L$. If $f \in M_a^\perp \cap E_1$ and $g \in N_b$ then

$$\begin{aligned} d_F(g + Tf, L) &= d_F(Tf, L) \\ &= \sup\{\Phi(Tf) : \Phi \in (L^\perp)_1 \subseteq F^*\} \\ &= \sup\{(T^*\Phi)(f) : \Phi \in (L^\perp)_1\}. \end{aligned}$$

Now, by taking the supremu over all $f \in M_a^\perp \cap E_1$ and $g \in N_b$, and by appealing to the Proposition, we obtain the equalities

$$\begin{aligned} \delta_F(N_b + T(M_a^\perp \cap E_1), L) &= \sup\{\|T^*\Phi \mid (M_a^\perp \cap E)\| : \Phi \in (L^\perp)_1\} \\ &= \sup\{d(T_K\varphi, M_a) : \varphi \in (L^\perp)_1 \subseteq L^\infty\}. \end{aligned}$$

Now by the Hobby–Rice theorem (Remark 2.2.2(1)) there exists $\xi \in \Lambda_n^-$ such that $h_\xi \in L^\perp \subseteq N_b^\perp \cap L^\infty$. Therefore

$$\delta_F(N_b + T(M_a^\perp \cap E_1), L) \geqslant e_n(\mathscr{K}, q).$$

The conclusion of the theorem now follows.

2.3. *Consequences of the Total Positivity Conditions*

This section is mainly concerned with extensions to well-known results in the theory of total positivity. The arguments are basically those of [14]. The extensions achieve some gain in efficiency. Pairs of theorems are replaced by single theorems. In particular Theorem 2.3.6 includes both [14, Lemma 3.1] and [14, Lemma 3.2]. The discussion also applies with trivial changes (specified in Section 3) to the periodic situation. The final Theorem 2.3.8 is independent of the rest of the section.

A basic procedure in the theory of total positivity involves the approximation of totally positive kernels by strictly totally positive ones. In the present situation the procedure requires an extension of "the basic composition formula" [6].

If $\mathscr{K} = (K; k_1,..., k_a; g_1,..., g_b)$ is a system and G is a kernel we define systems $\mathscr{K} * G$ and $G * \mathscr{K}$ by

$$\mathscr{K} * G = (K * G; k_1,..., k_a; g_1 * G,..., g_b * G),$$
$$G * \mathscr{K} = (G * K; G * k_1,..., G * k_a; g_1,..., g_b).$$

2.3.1. *Composition formulae*

$$(\mathscr{K} * G)\begin{pmatrix}1,..., b; & \xi_1,..., \xi_\rho\\ 1,..., a; & \tau_1,..., \tau_\sigma\end{pmatrix}$$
$$= \int_{\zeta_1<\zeta_2<\cdots<\zeta_\sigma}\cdots\int \mathscr{K}\begin{pmatrix}1,..., b; & \xi_1,..., \xi_\rho\\ 1,..., a; & \zeta_1,..., \zeta_\sigma\end{pmatrix} G\begin{pmatrix}\zeta_1,..., \zeta_\sigma\\ \tau_1,..., \tau_\sigma\end{pmatrix} d\zeta_1 \cdots d\zeta_\sigma,$$
$$(G * \mathscr{K})\begin{pmatrix}1,..., b; & \xi_1,..., \xi_\rho\\ 1,..., a; & \tau_1,..., \tau_\sigma\end{pmatrix}$$
$$= \int_{\zeta_1<\zeta_2<\cdots<\zeta_\rho}\cdots\int G\begin{pmatrix}\xi_1,..., \xi_\rho\\ \zeta_1,..., \zeta_\rho\end{pmatrix} \mathscr{K}\begin{pmatrix}1,..., b; & \zeta_1,..., \zeta_\rho\\ 1,..., a; & \tau_1,..., \tau_\sigma\end{pmatrix} d\zeta_1 \cdots d\zeta_\rho.$$

Proof. If $a = 0$ then the first identity is contained in the basic composition formula. The proof of the first identity involves three steps. (1) Replace the determinant

$$\mathscr{K}\begin{pmatrix}1,..., b; & \xi_1,..., \xi_\rho\\ 1,..., a; & \zeta_1,..., \zeta_\sigma\end{pmatrix}$$

on the right by its Laplace expansion by its first a columns. (2) Interchange sum and integral. (3) Apply the basic composition formula (the case $a = 0$) to each term of the sum. The resulting sum is the Laplace expansion of the left-hand side by its first a columns.

The second identity can be obtained by transposition from the first.

We can now introduce what is sometimes described as the "smoothing process" (see [6, p. 103]).

2.3.2. *Notation.* Let G_η, $\eta \neq 0$, be the kernel defined by

$$G_\eta(s, t) = \frac{1}{|\eta| \sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{s-t}{\eta}\right)^2\right).$$

For any system $\mathscr{K} = (K; k_1, ..., k_a; g_1, ..., g_b)$ let

$$K^{(\eta)} = G_\eta * (K * G_\eta),$$

$$\mathscr{K}^{(\eta)} = G_\eta * (\mathscr{K} * G_\eta).$$

For each $\eta \neq 0$ the kernel G_η is strictly totally positive (see [6]) and $(G_\eta, \eta \to 0)$ forms an approximate identity (in a sense to be made precise) for the algebra of continuous kernels. The properties of G_η which will be required will be listed in the following catch-all theorem.

2.3.3. THEOREM. (i) *For $\eta \neq 0$ the mapping $\eta \to G_\eta \in C([0, 1] \times [0, 1])$ is continuous with respect to uniform convergence of continuous kernels.*

(ii) *If $f \in C([0, 1])$ then $\|G_\eta * f\|_\infty \leqslant \|f\|_\infty$ and $(G_\eta * f)(s) \to f(s)$ as $\eta \to 0$ uniformly for s in each interval $[\delta, 1-\delta]$ with $\delta > 0$. If $s = 0$ or 1 then $(G_\eta * f)(s) \to \frac{1}{2}f(s)$ as $\eta \to 0$.*

(iii) *If $f \in C([0, 1])$ and $\varepsilon > 0$ then there exists $\eta_0 > 0$ such that*

$$\min\{f(s), 0\} - \varepsilon \leqslant (G_\eta * f)(s) \leqslant \max\{f(s), 0\} + \varepsilon$$

for all $s \in [0, 1]$ and all $\eta \in (0, \eta_0)$.

(iv) *If $K \in C([0, 1] \times [0, 1])$ then $K * G_\eta \to K$ as $\eta \to 0$ uniformly on each rectangle $[0, 1] \times [\delta, 1-\delta]$ with $\delta > 0$. The kernel $K^{(\eta)}$ converges uniformly to K as $\eta \to 0$ on each square $[\delta, 1-\delta] \times [\delta, 1-\delta]$ with $\delta > 0$.*

(v) *If the system $\mathscr{K}$ satisfies $(C1(a+\sigma))$ and $(C6(a+\sigma))$ then for each $\eta \neq 0$ the system $\mathscr{K}^{(\eta)}$ satisfies* (Strict $C1(a+\sigma)$).

Proof. Property (i) is obvious. Property (ii) is well-known and the proofs of (iii) and (iv) are similar. Property (v) is an immediate consequence of the composition formula 2.3.1 and the strict total positivity of the kernels G_η.

The point of (iii) is that it provides information about $G_\eta * f$ in the neighbourhoods of 0 and 1. It will be used in the proof of Theorem 2.4.5.

Integral operators with totally positive kernels have "variation diminishing" properties (see [6]). The next target is a theorem (2.3.6) of variation diminishing type. A lemma is needed—in the case $a = 0$ it is well known.

2.3.4. LEMMA. *Suppose that $\mathscr{K}$ satisfies $(C1(a+\sigma+1))$ and $(C6(a+\sigma+1))$.*

Let $\tau \in \Lambda_\sigma$ *and* $0 \leqslant \xi_1 < \cdots < \xi_{\rho+1} \leqslant 1$, *where* $\rho - a = \sigma - b \geqslant 0$. *Let* i^* *be one of* $1,\dots,\rho+1$ *and suppose that*

$$\operatorname{sgn} \mathscr{K}^{(\eta)} \begin{pmatrix} 1,\dots,b;\ \xi_1,\dots,\hat{\xi}_{i^*},\dots,\xi_{\rho+1} \\ 1,\dots,a;\ \tau_1,\dots,\tau_\sigma \end{pmatrix} = \varepsilon$$

for some η *in each interval* $(0, \delta)$, $\delta > 0$. *If* $\mathscr{K}$ *satisfies* $(C1(a+\sigma))$ *and* $(C6(a+\sigma))$ *then necessarily, by* 2.3.3(iv), $\varepsilon = 1$.

Then there exists a function φ *of the form*

$$\varphi = \sum_{i=1}^{b} \alpha_i g_i + \sum_{i=1}^{\rho+1} \beta_i K(\xi_i, \cdot)$$

such that

(i) *the coefficients* $\alpha_1,\dots,\alpha_b$, $\beta_1,\dots,\beta_{\rho+1}$ *are not all zero,*

(ii) $\sum_{i=1}^{\rho+1} \beta_i k_j(\xi_i) = 0$ *for* $j = 1,\dots,a$,

(iii) $(-1)^\sigma \varphi h_\tau \geqslant 0$,

(iv) $\beta_{i^*} = 0$ *if* $\varepsilon = 0$, $(-1)^{a+b+\sigma+i^*+1} \beta_{i^*} \varepsilon \geqslant 0$ *if* $\varepsilon = 1$ *or* -1.

Proof. Suppose first that $0 < \xi_1, \xi_{\rho+1} < 1$. Let φ_η be a function defined by

$$\begin{aligned} \varphi_\eta(t) &= \lambda \mathscr{K}^{(\eta)} \begin{pmatrix} 1,\dots,b;\ \xi_1,\dots,\xi_{\rho+1} \\ 1,\dots,a;\ \tau_1,\dots,\tau_\sigma, t \end{pmatrix} \\ &= \sum_{i=1}^{b} \alpha_i(\eta)\, g_i^{(\eta)}(t) + \sum_{i=1}^{\rho+1} \beta_i(\eta)\, K^{(\eta)}(\xi_i, t), \end{aligned}$$

where the sum denotes λ times the expansion of the determinant by its last column. By Theorem 2.3.3(v), for each $\eta \neq 0$ and $\lambda > 0$ the function φ_η is non-zero. Let $\lambda > 0$ be chosen so that

$$\max\{|\alpha_1(\eta)|,\dots,|\alpha_b(\eta)|, |\beta_1(\eta)|,\dots,|\beta_{\rho+1}(\eta)|\} = 1.$$

The function φ_η satisfies appropriate forms of (ii) and (iii). If the final column of the determinant is replaced by the jth column then expansion by the final column gives the appropriate form of (ii). The system $\mathscr{K}^{(\eta)}$ satisfies (Strict $C1(a+\sigma+1)$) (by 2.3.3(v)) and the appropriate form of (iii) follows immediately. The coefficient $\beta_{i^*}(\eta)$ is given by

$$\beta_{i^*}(\eta) = \lambda \mathscr{K}^{(\eta)} \begin{pmatrix} 1,\dots,b;\ \xi_1,\dots,\hat{\xi}_{i^*},\dots,\xi_{\rho+1} \\ 1,\dots,a;\ \tau_1,\dots\dots\dots,\tau_\sigma \end{pmatrix}.$$

We will let $\eta \to 0$ through a sequence such that $\operatorname{sgn} \beta_{i^*}(\eta) = \varepsilon$. Now let $(\alpha_1,..., \alpha_b, \beta_1,..., \beta_{\rho+1})$ be a cluster point of the sequence $(\alpha_1(\eta),..., \alpha_b(\eta), \beta_1(\eta),..., \beta_{\rho+1}(\eta))$. Then, passing to the limit and using 2.3.3(ii) and (iv), and the continuity of the functions, it follows that

$$\varphi = \sum_{i=1}^{b} \alpha_i g_i + \sum_{i=1}^{\rho+1} \beta_i K(\xi_i, \cdot)$$

satisfies (i)–(iv).

If $0 = \xi_1$ or $\xi_{\rho+1} = 1$ then replace ξ_1 by $\xi_1' > 0$ and $\xi_{\rho+1}$ by $\xi_{\rho+1}' < 1$. Apply the result in the case $0 < \xi_1' < \xi_2 < \cdots < \xi_\rho < \xi_{\rho+1}' < 1$ and repeat the process of taking a cluster point as $(\xi_1', \xi_{\rho+1}') \to (\xi_1, \xi_{\rho+1})$.

2.3.5. *Remark.* The lemma does not assert that $\varphi \neq 0$. However, if $\mathscr{K}$ satisfies $(C4'(a + \sigma + 1))$ and $\xi \in \Lambda_{\rho+1}$, or if $\mathscr{K}$ satisfies $(\text{Ext } C4'(a + \sigma + 1))$ and $0 < \xi_1$ then $\varphi \neq 0$.

The number of sign changes of a function f defined on $[0, 1]$ is denoted by $S^-(f)$. That is,

$$S^-(f) = \sup\{\rho\text{: there exist } 0 \leqslant \xi_1 < \cdots < \xi_{\rho+1} \leqslant 1 \text{ such that}$$
$$f(\xi_i) f(\xi_{i+1}) < 0 \text{ for } i = 1,..., \rho\},$$

and $S^-(f)$ either is a non-negative integer or is infinity. If $\tau \in \Lambda_\sigma$ then $S^-(h_\tau) = \sigma$.

The next theorem takes two forms according as $\mathscr{K}$ satisfies $(C4'(a + \sigma + 1))$ or $(\text{Ext } C4'(a + \sigma + 1))$. The second form is indicated by elements in square brackets.

2.3.6. THEOREM. *Suppose that $\mathscr{K}$ satisfies $(C1(a + \sigma + 1))$, $(C6(a + \sigma + 1))$ and $(C4'(a + \sigma + 1))$ [or $(\text{Ext } C4'(a + \sigma + 1))$], where $\sigma \geqslant b$. Let $\tau \in \Lambda_\sigma$ and*

$$u = k_0 + K * f + \sum_{j=1}^{\sigma} \kappa_j K(\cdot, \tau_j),$$

where $k_0 \in M_a$, f is integrable, $fh_\tau \geqslant 0$, $f \in N_b^\perp$ and

$$\sum_{j=1}^{\sigma} \kappa_j g(\tau_j) = 0 \qquad \textit{for all} \quad g \in N_b.$$

(i) *If $f^{-1}(0)$ is a Lebesgue null set then u has at most $\rho = a + \sigma - b$ zeros in $(0, 1)$ [or in $(0, 1]$]. If u has zeros at $0 < \xi_1 < \cdots < \xi_\rho < 1$ [or $\xi_\rho \leqslant 1$] then, for either $\varepsilon = 1$ or $\varepsilon = -1$, $\varepsilon u h_\xi \geqslant 0$. If $\mathscr{K}$ satisfies $(C1(a + \sigma))$ and $(C6(a + \sigma))$ then $\varepsilon = (-1)^{a+b}$.*

(ii) *If $\mathscr{K}$ satisfies (C3) then $S^-(u) \leqslant \rho = a + \sigma - b$.*

Proof. (i) Suppose that $f^{-1}(0)$ is null. Suppose that u has zeros at $0<\xi_1<\cdots<\xi_\rho<1$ [or $\xi_\rho\leqslant 1$]. Consider $\xi_*\in(0,1)\backslash\{\xi_1,\ldots,\xi_\rho\}$. Then $\xi_{i^*-1}<\xi_*<\xi_{i^*}$ for some i^* (where we interpret $\xi_0=0$ and $\xi_{\rho+1}=1$ if $\xi_\rho<1$). Put $S=\{\xi_1,\ldots,\xi_\rho,\xi_*\}$.

Now suppose that

$$\varphi=g_0+\sum_{s\in S}\beta_s K(s,\cdot)$$

has been chosen according to Lemma 2.3.4 (and Remark 2.3.5) so that $\varphi\neq 0$, $g_0\in N_b$, $\sum_{s\in S}\beta_s k(s)=0$ for $k\in M_a$ and $(-1)^\sigma\varphi h_\tau\geqslant 0$. We now have, using the facts that φ is continuous and $\neq 0$, $(-1)^\sigma\varphi f\geqslant 0$ and $f^{-1}(0)$ is null, that

$$(-1)^\sigma\beta_{\xi_*}u(\xi_*)=(-1)^\sigma\sum_{s\in S}\beta_s u(s)$$

$$=(-1)^\sigma\int\varphi(t)\,f(t)\,dt$$

$$>0.$$

This proves that $u(\xi_*)\neq 0$. However, it also shows that the sign of $\beta_{\xi_*}\neq 0$ was determined by u and that in the appeal to Lemma 2.3.4 there was no choice. Consequently

$$\operatorname{sgn}\mathscr{K}^{(\eta)}\begin{pmatrix}1,\ldots,b;\ \xi_1,\ldots,\xi_\rho\\ 1,\ldots,a;\ \tau_1,\ldots,\tau_\sigma\end{pmatrix}=\varepsilon'\neq 0$$

is contant for η in some interval $(0,\delta)$ and

$$(-1)^{a+b+\sigma+i^*+1}\beta_{\xi_*}\varepsilon'>0.$$

Now ε' is independent of ξ_*. This proves that $(-1)^{a+b}\varepsilon' h_\xi u\geqslant 0$. This completes the proof of (i).

(ii) Let $A=\{\varphi\in L^\infty([0,1])\colon\ \varphi h_\tau\geqslant 0,\ \inf|\varphi(t)|>0\}$. Then A is convex and open in $L^\infty([0,1])$. We will show that $A\cap N_b^\perp\neq\varnothing$. For suppose on the contrary that $A\cap N_b^\perp=\varnothing$. Then there exists a linear functional $\Phi\in L^\infty([0,1])^*$ which separates A and $N_b^\perp$: $\Phi(\varphi)=0$ for all $\varphi\in N_b^\perp$ and $\Phi(\varphi)>0$ for all $\varphi\in A$. It follows from the first of these conditions that there exists $g_0\in N_b$ such that $\Phi(\varphi)=\langle g_0,\varphi\rangle$ for all $\varphi\in L^\infty([0,1])$. Now by condition $(C3)$ the function g_0 has at most $b-1$ zeros in $(0,1)$ and $b-1<\sigma$. Therefore $g_0 h_\tau$ is not of constant sign. We now obtain a contradiction, for we can easily show (using the continuity of g_0) that $\Phi(\varphi)=\langle g_0,\varphi\rangle<0$ for some $\varphi\in A$.

Now suppose that $\varphi \in N_b^\perp \cap A$. The conclusion (ii) follows by applying (i) to $f + \mu\varphi$ ($\mu > 0$) and letting $\mu \to 0$.

The remainder of this section is concerned with the kernels H_0 (of Theorem 2.1.5) which are best approximations to K in the sense of $a_n(\mathscr{K}, |\cdot|_{\infty,q})$. Lemma 2.3.7 establishes one clause of Theorem 2.1.4 and ensures that the kernels H_0 are defined. The proof of Lemma 2.3.7 follows that of [12, Lemma 7.2]. Theorem 2.3.8 describes some of the properties of the kernels H_0 and shows that under suitable assumptions they are indeed candidates for best approximations to K in the sense of $a_n(\mathscr{K}; |\cdot|_{\infty,q})$. It is this last fact for which we require that $\mathscr{K}$ satisfies $(C2'(a+\sigma))$.

2.3.7. LEMMA. *Suppose that* $\mathscr{K}$ *satisfies* $(C1(a+\sigma+1))$, $(C6(a+\sigma+1))$, $(C4(a+\sigma))$ *and* $(C4'(a+\sigma+1))$.

Suppose that a function $P_0 = k_0 + K * h_{\tau^0}$, *where* $k_0 \in M_a$, $\tau^0 \in \Lambda_\sigma$ *and* $h_{\tau^0} \in N_b^\perp$, *has zeros at the point of* $\xi^0 \in \Lambda_\rho$, $\rho = a + \sigma - b$. *Then*

$$\mathscr{K}\begin{pmatrix} 1,\ldots, b;\ \xi_1^0,\ldots, \xi_\rho^0 \\ 1,\ldots, a;\ \tau_1^0,\ldots, \tau_\sigma^0 \end{pmatrix} \neq 0.$$

Proof. Suppose not. Then there exist $\alpha_1,\ldots,\alpha_a$, $\kappa_1,\ldots,\kappa_\sigma$, not all zero such that

$$\sum_{j=1}^{\sigma} \kappa_j g(\tau_j^0) = 0 \qquad \text{for all} \quad g \in N_b$$

and

$$\sum_{j=1}^{a} \alpha_j k_j(\xi_i^0) + \sum_{j=1}^{\sigma} \kappa_j K(\xi_i^0, \tau_j^0) = 0 \qquad \text{for} \quad i = 1,\ldots, \rho.$$

Now, by $(C4(a+\sigma))$, for some $\xi_* \in (0, 1)$

$$\sum_{j=1}^{a} \alpha_j k_j(\xi_*) + \sum_{j=1}^{\sigma} \kappa_j K(\xi_*, \tau_j^0) \neq 0.$$

Then we can choose λ so that

$$P_0(\xi_*) + \lambda \left(\sum_{j=1}^{a} \alpha_j k_j(\xi_*) + \sum_{j=1}^{\sigma} \kappa_j K(\xi_*, \tau_j^0)\right) = 0.$$

Then $P_0 + \lambda(\sum_{j=1}^{a} \alpha_j k_j + \sum_{j=1}^{\sigma} \kappa_j K(\cdot, \tau_j^0))$ has $\rho + 1$ zeros in $(0, 1)$. This contradicts Theorem 2.3.6(i).

2.3.8. THEOREM. *Suppose that* $\mathscr{K}$ *satisfies* (C3) *and* (C2'$(a+\sigma)$). *Suppose that* $\tau^0 \in \Lambda_\sigma$ *and* $\xi^0 \in \Lambda_\rho$ *are such that*

$$\mathscr{K}\begin{pmatrix} 1,..., b; & \xi_1^0,..., \xi_\rho^0 \\ 1,..., a; & \tau_1^0,..., \tau_\sigma^0 \end{pmatrix} \neq 0.$$

Let

$$H_0(s,t) = K(s,t) - \frac{\mathscr{K}\begin{pmatrix} 1,..., b; & \xi_1^0,..., \xi_\rho^0, s \\ 1,..., a; & \tau_1^0,..., \tau_\sigma^0, t \end{pmatrix}}{\mathscr{K}\begin{pmatrix} 1,..., b; & \xi_0^1,..., \xi_\rho^0 \\ 1,..., a; & \tau_1^0,..., \tau_\sigma^0 \end{pmatrix}},$$

and let

$$L_0 = \left\{ \sum_{j=1}^{a} \alpha_j k_j + \sum_{j=1}^{\sigma} \beta_j K(\cdot, \tau_j) : \sum_{j=1}^{\sigma} \beta_j g(\tau_j) = 0 \text{ for } g \in N_b \right\}.$$

Then (i) $\dim L_0 = \rho$.

(ii) $H_0 * (N_b^\perp \cap C([0,1])) = L_0 \supseteq M_a$.

(iii) L_0 *interpolates at* $\xi_1^0,..., \xi_\rho^0$. *If* $T_0: C([0,1]) \to L_0$ *is the operator of interpolation at* $\xi_1^0,..., \xi_\rho^0$ *then* $T_{H_0} \mid N_b^\perp = T_0(T_K \mid N_b^\perp)$.

(iv) *If* $\mathscr{K}$ *also satisfies* (C1$(a+\sigma+1)$), $h_{\tau^0} \in N_b^\perp$ *and* $P_0 = k_0 + K * h_{\tau^0}$, *where* $k_0 \in M_a$, *is a function which is zero at the points of* ξ^0 *then*

$$|K - H_0|_{\infty, q} = \|P_0\|_q.$$

Proof. If $f \in N_b^\perp$ then

$$\int \mathscr{K}\begin{pmatrix} 1,..., b; & \xi_1^0,..., \xi_\rho^0, s \\ 1,..., a; & \tau_1^0,..., \tau_\sigma^0, t \end{pmatrix} f(t)\,dt$$

$$= \begin{vmatrix} 0 & \cdots & 0 & g_1(\tau_1^0) & \cdots & g_1(\tau_\sigma^0) & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & g_b(\tau_1^0) & \cdots & g_b(\tau_\sigma^0) & 0 \\ k_1(\xi_1^0) & \cdots & k_a(\xi_1^0) & K(\xi_1^0, \tau_1^0) & \cdots & K(\xi_1^0, \tau_\sigma^0) & (K*f)(\xi_1^0) \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ k_1(\xi_\rho^0) & \cdots & k_a(\xi_\rho^0) & K(\xi_\rho^0, \tau_1^0) & \cdots & K(\xi_\rho^0, \tau_\sigma^0) & (K*f)(\xi_\rho^0) \\ k_1(s) & \cdots & k_a(s) & K(s, \tau_1^0) & \cdots & K(s, \tau_\sigma^0) & (K*f)(s) \end{vmatrix}$$

$$= \sum_{j=1}^{a} \alpha_j k_j(s) + \sum_{j=1}^{\sigma} \beta_j K(s, \tau_j^0) + \mathscr{K}\begin{pmatrix} 1,..., b; \xi_1^0,..., \xi_\rho^0 \\ 1,..., a; \tau_1^0,..., \tau_\sigma^0 \end{pmatrix} \cdot (K*f)(s)$$

for some $\alpha_1,\dots,\alpha_a$, $\beta_1,\dots,\beta_\sigma$ satisfying the equation

$$\sum_{j=1}^{\sigma} \beta_j g(\tau_j^0) = 0 \qquad \text{for all} \quad g \in N_b^\perp.$$

This shows that

$$H_0 * (N_b^\perp \cap C([0,1])) \subseteq L_0.$$

It is an immediate consequence of the definition of H_0 that $(H_0 * f)(\xi_i^0) = (K * f)(\xi_i^0)$ for all $i = 1,\dots,\rho$ and all $f \in L^1([0,1])$. Consider the mapping φ: $C([0,1]) \to \mathbb{R}^{b+\rho}$ defined by

$$\varphi(f) = (\langle f, g_1\rangle,\dots,\langle f, g_b\rangle, (K * f)(\xi_1^0),\dots,(K * f)(\xi_\rho^0)).$$

For each $t \in [0,1]$ the point $(g_1(t),\dots,g_b(t), K(\xi_1^0, t),\dots,K(\xi_\rho^0, t))$ is in the closure of $\varphi(C([0,1]))$ (let f "tend" to the unit point measure at t) and is therefore in $\varphi(C([0,1]))$. Now, by $(C2'(b+\rho = a+\sigma))$, $\varphi(C([0,1])) = \mathbb{R}^{b+\rho}$. It now follows that the composite mapping

$$N_b^\perp \cap C([0,1]) \xrightarrow{f \to K * f} C([0,1]) \xrightarrow{f \to (f(\xi_1^0),\dots,f(\xi_\rho^0))} \mathbb{R}^\rho$$

is surjective. But it coincides with the composite

$$N_b^\perp \cap C([0,1]) \xrightarrow{f \to H_0 * f} C([0,1]) \xrightarrow{f \to (f(\xi_1^0),\dots,f(\xi_\rho^0))} \mathbb{R}^\rho.$$

Consequently $\dim H_0 * (N_b^\perp \cap C([0,1])) \geqslant \rho$. However, by $(C3)$ and the definition of L_0, $\dim L_0 \leqslant \rho$. This proves (i), (ii) and (iii).

If $\mathscr{K}$ satisfies $(C1(a+\sigma+1))$ then

$$\int \left| \mathscr{K}\begin{pmatrix} 1,\dots,b; & \xi_1^0,\dots,\xi_\rho^0, s \\ 1,\dots,a; & \tau_1^0,\dots,\tau_\sigma^0, t \end{pmatrix} \right| dt = \pm \int \mathscr{K}\begin{pmatrix} 1,\dots,b; & \xi_1^0,\dots,\xi_\rho^0, s \\ 1,\dots,a; & \tau_1^0,\dots,\tau_\sigma^0, t \end{pmatrix} h_{\tau^0}(t)\, dt.$$

If $(K * h_{\tau^0})(\xi_i^0) = P_0(\xi_i^0) - k_0(\xi_i^0) = -k_0(\xi_i^0)$ for $i = 1,\dots,\rho$ then by the calculation at the beginning of the proof, using the fact that $k_0 \in M_a = \operatorname{sp}\{k_1,\dots,k_a\}$,

$$\int \mathscr{K}\begin{pmatrix} 1,\dots,b; & \xi_1^0,\dots,\xi_\rho^0, s \\ 1,\dots,a; & \tau_1^0,\dots,\tau_\sigma^0, t \end{pmatrix} h_{\tau^0}(t)\, dt = \mathscr{K}\begin{pmatrix} 1,\dots,b; & \xi_1^0,\dots,\xi_\rho^0 \\ 1,\dots,a; & \tau_1^0,\dots,\tau_\sigma^0 \end{pmatrix} \cdot P_0(s).$$

This proves (iv).

2.4. *The Variational Problem* $e_m(\mathscr{K}, q)$

Recall that, for $m \geqslant b$,

$$e_m(\mathscr{K}, q) = \inf\{\|k + K * h_\tau\|_q : k \in M_a, \tau \in \Lambda_m^-, h_\tau \in N_b^\perp\}.$$

We will require that $\mathscr{K}$ satisfies (C3), and $g_1,..., g_b$ will therefore form a weak Chebyshev system. It is then a consequence that if $\tau \in \Lambda_\sigma$ and $h_\tau \in N_b^\perp$ then $\sigma \geqslant b$. By the Hobby–Rice theorem the set $\{\tau \in \Lambda_m^- : h_\tau \in N_b^\perp\}$ is non-empty if $m \geqslant b$; it is also a closed subset of the compact set Λ_m^-. The function $\tau \to K * h_\tau \in L^q$ is continuous on Λ_m^-. It follows that there exists a function P_0 which is extremal for $e_m(\mathscr{K}, q)$. That is, P_0 is of the form

$$P_0 = k_0 + K * h_{\tau^0},$$

where $k_0 \in M_a$, $\tau^0 \in \Lambda_\sigma$ and $\sigma \leqslant m$, $h_{\tau^0} \in N_b^\perp$ and

$$\|P_0\|_q = e_m(\mathscr{K}, q).$$

Conditions which must be satisfied by P_0 will be derived by considering variations of P_0. The argument, initially, follows that of Tikhomirov [21]. The variations are of two kinds: variation of the points of τ^0 and, if $\sigma < m$, the extension of τ^0 by the addition of points. There are three cases to consider. The notation will be chosen to cover all cases simultaneously.

Case (1). If $\sigma \leqslant m - 2$ then we may add two points to τ^0. In this case let $\tau^0_{\sigma+1}$ be a point with $\tau^0_\sigma < \tau^0_{\sigma+1} < 1$ and let A denote $(-\infty, 0]$.

Case (2). If $\sigma = m - 1$ then we may add one point to τ^0. In this case let $\tau^0_{\sigma+1} = 1$ and again let A denote $(-\infty, 0]$.

Case (3). If $\sigma = m$ then we cannot add additional points to τ^0. In this case let $\tau^0_{\sigma+1} = 1$ but let $A = \{0\}$.

The next lemma is a development of [21, Proposition 2]. Let $B(0, r)$ denote the open ball $\{f \in L^q : \|f\|_q < r\}$ in L^q.

2.4.1. LEMMA. *Suppose that condition* (C3) *is satisfied. Let*

$$V = \left\{k + \sum_{j=1}^{\sigma+1} u_j K(\cdot, \tau_j^0) : k \in M_a, (u_1,..., u_{\sigma+1}) \in \mathbb{R}^{\sigma+1}, (-1)^\sigma u_{\sigma+1} \in A, \right.$$
$$\left. \sum_{j=1}^{\sigma+1} u_j g(\tau_j^0) = 0 \text{ for } g \in N_b \right\}.$$

Then

$$B(0, \|P_0\|_q) \cap (P_0 + V) = \varnothing.$$

Proof. Suppose that

$$p = \bar{k} + \sum_{j=1}^{\sigma+1} 2(-1)^{j-1} \bar{u}_j K(\cdot, \tau_j^0) \in V$$

(so that $\bar{u}_{\sigma+1} \in A$) and

$$\| P_0 + p \| = \|P_0\| - \delta_0 < \|P_0\|.$$

We will obtain a contradiction to the fact that P_0 is extremal for $e_m(\mathscr{K}, q)$.

The proof will involve an appeal to the implicit function theorem which is usually stated in terms of functions defined on open sets. In order to invoke the theorem it is convenient to extend the domains of definition of K and $g_1, \ldots, g_b$ by

$$K(s, t) = K(s, 1), \qquad g_i(t) = g_i(1)$$

for $t \geqslant 1$, all $s \in [0, 1]$ and $i = 1, \ldots, b$.

If $\bar{u}_{\sigma+1} = 0$ let $\nu = \sigma$ and $\bar{\tau}^0 = \tau^0$. If $\bar{u}_{\sigma+1} \neq 0$ (and so $\bar{u}_{\sigma+1} < 0$) let $\nu = \sigma + 1$ and $\bar{\tau}^0 = (\tau_1^0, \ldots, \tau_\tau^0, \tau_{\sigma+1}^0)$. Thus, in both these cases $\bar{\tau}^0 \in \Lambda_\nu^- \subseteq \mathbb{R}^\nu$.

Let $W = \{\tau_1, \ldots, \tau_\nu) \in \mathbb{R}^\nu\colon 0 < \tau_1 < \cdots < \tau_\nu\}$. Then W is a neighbourhood of $\bar{\tau}^0$. Define mappings

$$\theta\colon W \longrightarrow \mathbb{R}^b,$$

$$\varphi\colon M_a \times W \longrightarrow L^q$$

by

$$\theta(\tau)_i = \sum_{j=1}^{\nu} 2(-1)^{j-1} \int_0^{\tau_j} g_i(t)\, dt + \varepsilon_\nu \cdot 2(-1)^{\sigma+1} \int_0^{\tau_{\sigma+1}^0} g_i(t)\, dt$$
$$+ (-1)^{\sigma+2} \int_0^1 g_i(t)\, dt,$$

$$\varphi(k, \tau)(s) = k(s) + \sum_{j=1}^{\nu} 2(-1)^{j-1} \int_0^{\tau_j} K(s, t)\, dt$$
$$+ \varepsilon_\nu \cdot 2(-1)^{\sigma+1} \int_0^{\tau_{\sigma+1}^0} K(s, t)\, dt + (-1)^{\sigma+2} \int_0^1 K(s, t)\, dt,$$

where $\varepsilon_\nu = 1$ if $\nu = \sigma + 1$ and $\varepsilon_\nu = 0$ if $\nu = \sigma$.

If $\tau = (\tau_1, \ldots, \tau_\nu)$ and $0 < \tau_1 < \cdots < \tau_\nu < \tau_{\sigma+1}^0$ let

$$\begin{aligned} \tau' &= (\tau_1, \ldots, \tau_\sigma, \tau_{\sigma+1}, \tau_{\sigma+1}^0) \in \Lambda_{\sigma+2}, && \text{in Case (1) if } \nu = \sigma + 1, \\ &= (\tau_1, \ldots, \tau_\sigma, \tau_{\sigma+1}) \in \Lambda_{\sigma+1}, && \text{in Case (2) if } \nu = \sigma + 1, \\ &= (\tau_1, \ldots, \tau_\sigma) \in \Lambda_\sigma, && \text{if } \nu = \sigma. \end{aligned}$$

Then

$$\theta(\tau)_i = \langle g_i, h_{\tau'} \rangle, \qquad \varphi(k, \tau) = k + K * h_{\tau'}.$$

Note that these equations do not hold if $v = \sigma + 1$ and $\tau^0_{\sigma+1} < \tau_{\sigma+1}$. Furthermore

$$\theta(\bar{\tau}^0) = 0, \qquad \varphi(k_0, \bar{\tau}^0) = P_0.$$

A straightforward calculation (appealing to the continuity of K and $g_1, ..., g_b$) shows that θ and φ are Frechet differentiable mappings and that the derivatives are given by

$$\theta'(\tau)(u)_i = \sum_{j=1}^{v} 2(-1)^{j-1} u_j g_i(\tau_j),$$

$$\varphi'(k_0, \bar{\tau}^0)(k, u) = k + \sum_{j=1}^{v} 2(-1)^{j-1} u_j K(\cdot, \tau_j^0).$$

Therefore

$$p = \varphi'(k_0, \bar{\tau}^0)(\bar{k}, \bar{u}),$$

where $\bar{k} \in M_a$ and $\bar{u} = (\bar{u}_1, ..., \bar{u}_v) \in \ker \theta'(\bar{\tau}^0)$.

The mapping

$$W \xrightarrow{\tau \to \theta'(\tau)} \mathscr{L}(\mathbb{R}^v, \mathbb{R}^b)$$

is continuous and, by (C3), $\theta'(\bar{\tau}^0)$: $\mathbb{R}^v \to \mathbb{R}^b$ is of rank b (recall that $\sigma \geqslant b$ and τ^0 contains the σ points $\tau_1^0, ..., \tau_\sigma^0$ of $(0, 1)$). Now by a routine applications of the implicit function theorem there is a neighbourhood W_0 of $\bar{\tau}^0 \in \mathbb{R}^v$ and a mapping

$$\psi \colon W_0 \cap (\bar{\tau}^0 + \ker \theta'(\bar{\tau}^0)) \to \theta^{-1}(0) \subset \mathbb{R}^v$$

such that $\psi(\bar{\tau}^0) = \bar{\tau}^0$, ψ is differentiable and $\psi'(\bar{\tau}^0)(u) = u$ for $u \in \ker \theta'(\bar{\tau}^0)$.

If $0 < \varepsilon < 1$ then

$$\|P_0 + \varphi'(k_0, \bar{\tau}^0)(\varepsilon\bar{k}, \varepsilon\bar{u})\| = \|P_0 + \varepsilon p\| \leqslant \|P_0\| - \varepsilon\delta_0.$$

A simple calculation now shows that for small $\varepsilon > 0$

$$\|\varphi(k_0 + \varepsilon\bar{k}, \psi(\bar{\tau}_0 + \varepsilon\bar{u}))\| < \|P_0\|.$$

However, $\|\psi(\bar{\tau}^0 + \varepsilon\bar{u}) - \bar{\tau}^0 - \varepsilon\bar{u}\| = \varepsilon \|\bar{u}\|.\ O(1)$ as $\varepsilon \to 0$. Therefore if $v = \sigma + 1$ and $\bar{u}_{\sigma+1} < 0$ then for small $\varepsilon > 0$ the coordinate

$(\psi(\bar{\tau}^0 + \varepsilon\bar{u}) - \bar{\tau}^0)_{\sigma+1}$ has the same sign as $\bar{u}_{\sigma+1}$. Therefore for small $\varepsilon > 0$, in all cases, $\varphi(k_0 + \varepsilon\bar{k}, \psi(\bar{\tau}_0 + \varepsilon\bar{u}))$ is of the form $k_0 + \varepsilon\bar{k} + K * h_{\tau'}$ for some $\tau' \in \Lambda_{\sigma'}$ with $\sigma' \leqslant m$ and $h_{\tau'} \in N_b^{\perp}$. This contradicts the extremal property of P_0. The proof of the lemma is complete.

The two cases $1 \leqslant q < \infty$ and $q = \infty$ are now discussed separately; the latter case in two stages, first assuming that (Strict $C1$) is satisfied, and then assuming only ($C1$).

2.4.2. THEOREM. *Suppose that $1 \leqslant q < \infty$ and that $m \geqslant b$. Suppose that $\mathscr{K}$ satisfies* ($C3$), ($C1(a+\nu)$), ($C4(a+\nu)$) *and* ($C4'(a+\nu)$) *for $b \leqslant \nu \leqslant m+1$.*

If $P_0 = k_0 + K * h_{\tau^0}$, where $k_0 \in M_a$, $\tau^0 \in \Lambda_\sigma$ and $\sigma \leqslant m$, and $h_{\tau^0} \in N_b^{\perp}$, is extremal for $e_m(\mathscr{K}, q)$ then

(i) *P_0 has precisely $\rho = a - b + \sigma$ zeros in* (0, 1);

(ii) *either $\sigma = m - 1$ or $\sigma = m$; if $\mathscr{K}$ also satisfies* (Ext $C4(a+m)$) *then $\sigma = m$. Futhermore*

(iii) *if $b + 2 \leqslant r \leqslant m$ then $e_r(\mathscr{K}, q) < e_{r-2}(\mathscr{K}, q)$; if $\mathscr{K}$ also satisfies* (Ext $C4(a+m)$) *and $b + 1 \leqslant r \leqslant m$ then $e_r(\mathscr{K}, q) < e_{r-1}(\mathscr{K}, q)$.*

Proof. Note that, by Proposition 2.1.2(ii), $\mathscr{K}$ satisfies ($C6(a+\nu)$) for $b \leqslant \nu \leqslant m+1$.

By Theorem 2.3.6 the function P_0 has at most $\rho = a - b + \sigma$ zeros in (0, 1). So $\|P_0\|_q \neq 0$. Let P_0 change sign at the points of $\xi^0 \in \Lambda_p$, where $p \leqslant \rho$. Choose $\varepsilon = \pm 1$ so that $\varepsilon P_0 h_{\xi^0} \geqslant 0$. Let

$$\varphi = \varepsilon \frac{|P_0|^{q-1}}{\|P_0\|^{q-1}} \operatorname{sgn} P_0.$$

Then $\varphi h_{\xi^0} \geqslant 0$ and $\varphi^{-1}(0)$ is a null set. The linear functional $\Phi \in (L^q)^*$ defined by

$$\Phi(f) = \int \varepsilon\varphi(t) f(t)\, dt$$

is the unique support functional to $B(0, \|P_0\|)$ at P_0 such that $\|\Phi\| = 1$ and $\Phi(P_0) = \|P_0\|$. (If $1 < q < \infty$ then the space L^q is smooth, if $q = 1$ then P_0 is a smooth point of the closed ball $B'(0, \|P_0\|)$ because $\varphi^{-1}(0)$ is null).

The set V of Lemma 2.4.1 is convex, and so, by Lemma 2.4.1, the linear functional Φ separates $B(0, \|P_0\|)$ and the convex set $P_0 + V$. Therefore

$$\int \varphi(t)\, k(t) = 0 \qquad \text{for all} \quad k \in M_a$$

and

$$\sum_{j=1}^{\sigma+1} \varepsilon u_j(\varphi * K)(\tau_j^0) \geqslant 0$$

whenever both

$$\sum_{j=1}^{\sigma+1} u_j g(\tau_j^0) = 0 \qquad \text{for all} \quad g \in N_b,$$

and

$$(-1)^\sigma u_{\sigma+1} \in A.$$

This condition, in the case that $u_{\sigma+1} = 0$, implies that for some $g_0 \in N_b$

$$(g_0 + \varphi * K)(\tau_j^0) = 0 \qquad \text{for} \quad j = 1, ..., \sigma.$$

Now by Theorem 2.3.6 applied to the transposed system $\mathscr{K}'$ (and to the function φ) it follows that $\sigma \leqslant b - a + p$, that is, $p \geqslant a - b + \sigma = \rho$. This proves that $p = a - b + \sigma$, and proves (i).

It now follows from Theorem 2.3.6 that $\varepsilon = (-1)^{a+b}$. Also, by Theorem 2.3.6 applied to $\mathscr{K}'$

$$(-1)^{a+b}(g_0 + \varphi * K) h_{\tau^0} \geqslant 0.$$

By condition $(C3)$ there exist $u_1, ..., u_{\sigma+1}$ such that

$$\sum_{j=1}^{\sigma+1} u_j g(\tau_j^0) = 0 \qquad \text{for all} \quad g \in N_b$$

and such that $u_{\sigma+1} \neq 0$. We can choose such coefficients with $(-1)^\sigma u_{\sigma+1} < 0$. If $\sigma \leqslant m - 2$ (Case 1) or $\sigma = m - 1$ (Case 2) then $A = (-\infty, 0]$. Now for these coefficients

$$\begin{aligned} 0 &\leqslant \sum_{j=1}^{\sigma+1} \varepsilon u_j(\varphi * K)(\tau_j^0) \\ &= \sum_{j=1}^{\sigma+1} \varepsilon u_j(g_0 + \varphi * K)(\tau_j^0) \\ &= \varepsilon u_{\sigma+1}(g_0 + \varphi * K)(\tau_{\sigma+1}^0) \\ &= -((-1)^{\sigma+1} u_{\sigma+1})((-1)^{a+b+\sigma}(g_0 + \varphi * K)(\tau_{\sigma+1}^0)) \end{aligned}$$

so that

$$(-1)^{a+b+\sigma}(g_0 + \varphi * K)(\tau_{\sigma+1}^0) = 0.$$

If we had $\sigma \leqslant m-2$ (Case 1) and $\tau_\sigma^0 < \tau_{\sigma+1}^0 < 1$ this would contradict Theorem 2.3.6. This proves that σ is either $m-1$ or m. If $\mathscr{K}$ satisfies (Ext $C4(a+m)$) and we had $\sigma = m-1$ (Case 2), so that $\tau_{\sigma+1}^0 = 1$, we would again have a contradiction to Theorem 2.3.6. This completes the proof of (ii).

It follows from (ii) that a function P_0' which is extremal for $e_{m-2}(\mathscr{K}, q)$ cannot be extremal for $e_m(\mathscr{K}, q)$. Therefore $e_{m-2}(\mathscr{K}, q) > e_m(\mathscr{K}, q)$. In the same way, if (Ext $C4(a+m)$) is satisfied then $e_{m-1}(\mathscr{K}, q) > e_m(\mathscr{K}, q)$. The conditions for the integer m contain the conditions for smaller integers. Therefore (iii) follows.

The next two theorems are concerned with the case $q = \infty$.

2.4.3. THEOREM. *Let $m \geqslant b$. Suppose that $\mathscr{K}$ satisfies* (Strict $C1(a+v)$) *for $b \leqslant v \leqslant m+1$.*

If $P_0 = k_0 + K * h_{\tau^0}$, where $k_0 \in M_a$, $\tau^0 \in \Lambda_\sigma$ and $\sigma \leqslant m$, and $h_{\tau^0} \in N_b^\perp$, is extremal for $e_m(\mathscr{K}, \infty)$ then

(i) *$\sigma = m$, P_0 has precisely $\rho = a - b + m$ zeros in* (0, 1) *and there exist $\rho + 1$ points of* [0, 1] *at which P_0 attains its bound $\|P_0\|_\infty$ with alternating signs;*

(ii) *if $b + 1 \leqslant r \leqslant m$ then $e_r(\mathscr{K}, \infty) < e_{r-1}(\mathscr{K}, \infty)$.*

The proof of the theorem requires a simple lemma. The implications of Proposition 2.1.2 will be used without comment.

2.4.4. LEMMA. *The subspace*

$$V' = \left\{ k + \sum_{j=1}^{\sigma} u_j K(\cdot, \tau_j^0) : k \in M_a, \sum_{j=1}^{\sigma} u_j g(\tau_j^0) = 0 \text{ for } g \in N_b \right\}$$

is a Chebyshev subspace of $C([0, 1])$.

Proof. V' is of dimension $\rho = a + \sigma - b$. It is easily seen that any function in V' which has ρ zeros is the zero function (by Strict $C1(a+\sigma)$)). The conclusion follows by Haar's theorem.

Proof of the theorem. It follows from Lemma 2.4.1 that 0 is a best approximation to P_0 from $V' \subseteq V \subseteq C([0, 1])$. Therefore, because V' is Chebyshev, P_0 attains its bound with alternating signs at $\rho + 1$ points of [0, 1], and has at least ρ zeros on (0, 1). But, by Theorem 2.3.6, P_0 has at most ρ zeros on (0, 1). Therefore P_0 has precisely ρ zeros on (0, 1). If $m = b$ there is nothing more to prove. If $m \geqslant b + 1$ and the zeros of P_0 are at $\xi^0 \in \Lambda_\rho$ then $(-1)^{a+b} P_0 h_{\xi^0} \geqslant 0$.

Now suppose that $\sigma < m$ (Case (1) or Case (2)). Let

$$\varphi(s) = (-1)^{\sigma+1}\mathscr{K}\begin{pmatrix} 1,..., b; \ \xi_1^0,..., \xi_\rho^0, s \\ 1,..., a; \ \tau_1^0,..., \tau_\sigma^0, \tau_{\sigma+1}^0 \end{pmatrix}.$$

Then

$$\varphi = \sum_{j=1}^{a} \alpha_j k_j + \sum_{j=1}^{\sigma+1} u_j K(\cdot, \tau_j^0),$$

where

$$\sum_{j=1}^{\sigma+1} u_j g(\tau_j^0) = 0 \qquad \text{for all} \quad g \in N_b$$

and

$$u_{\sigma+1} = (-1)^{\sigma+1}\mathscr{K}\begin{pmatrix} 1,..., b; \ \xi_1^0,..., \xi_\rho^0 \\ 1,..., a; \ \tau_1^0,..., \tau_\sigma^0 \end{pmatrix}.$$

Therefore $\varphi \in V$. By the condition (Strict $C1(a+\sigma+1)$) φ is zero only at the points ξ^0 and $(-1)^{a+b+1}\varphi h_{\xi^0} \geqslant 0$. It follows that, for small $\varepsilon > 0$, $\|P_0 + \varepsilon\varphi\|_\infty < \|P_0\|_\infty$, and this contradict, Lemma 2.4.1. This proves (i), and (ii) follows.

2.4.5. THEOREM. *Let $m \geqslant b$. Suppose that $\mathscr{K}$ satisfies (C3) and $(C1(a+v))$, $(C6(a+v))$ for $b \leqslant v \leqslant m+1$. Then there exists a function $P_0 = k_0 + K * h_{\tau^0}$, where $k_0 \in M_a$, $\tau_0 \in \Lambda_\sigma$ and $\sigma \leqslant m$, and $h_{\tau^0} \in N_b^\perp$, such that*

(i) $\|P_0\|_\infty = e_m(\mathscr{K}, \infty)$,

(ii) *there exist $a - b + m + 1$ points of $[0, 1]$ at which P_0 attains its bound $\|P_0\|_\infty$ with alternating signs.*

Proof. The proof will use the notation and conclusions of 2.3.2 and 2.3.3.

By Theorem 2.3.3(v), for each $\eta \neq 0$ the system $\mathscr{K}^{(\eta)}$ satisfies (Strict $C1(a+v)$) for $b \leqslant v \leqslant m+1$. Let

$$P_\eta = k_\eta + K^{(\eta)} * h_{\tau^{(\eta)}} = \sum_{j=1}^{a} \alpha_j(\eta) k_j^{(\eta)} + K^{(\eta)} * h_{\tau^{(\eta)}}$$

be extremal for $e_m(\mathscr{K}^{(\eta)}, \infty)$. Let $(\eta_r)_{r\geqslant 1}$ be a sequence tending to zero such that

$$\liminf_{\eta\to 0} e_m(\mathscr{K}^{(\eta)}, \infty) = \lim_{r\to\infty} \|P_{\eta_r}\|$$

and such that the limits

$$(\alpha_1,\ldots,\alpha_a)=\lim_{r\to\infty}(\alpha_1(\eta_r),\ldots,\alpha_a(\eta_r)),$$

$$\tau^0=\lim_{r\to\infty}\tau^{(\eta_r)}\in\Lambda_m^-$$

exist. Let

$$P_0=\sum_{j=1}^{a}\alpha_j k_j+K*h_{\tau^0}=k_0+K*h_{\tau^0}.$$

It will be shown that P_0 satisfies (i) and (ii). It follows easily from Theorem 2.3.3(ii) that $h_{\tau^0}\in N_b^\perp$. Another routine calculation, using Theorem 2.3.3(iv), shows that

$$\|P_{\eta_r}-G_{\eta_r}*P_0\|_\infty\to 0 \qquad \text{as} \quad r\to\infty,$$

and it follows from this that $\|P_0\|_\infty=\lim_{r\to\infty}\|P_{\eta_r}\|_\infty$.

Next it will be shown that

$$e_m(\mathscr{K},\infty)\geqslant\limsup_{\eta\to 0}e_m(\mathscr{K}^{(\eta)},\infty).$$

Let $\bar{P}=\bar{k}+K*h_{\bar{\tau}}$, where $\bar{k}\in M_a$, $\bar{\tau}\in\Lambda_\sigma$ and $\sigma\leqslant m$, and $h_{\bar{\tau}}\in N_b^\perp$, be extremal for $e_m(\mathscr{K},\infty)$. Consider the mapping

$$\psi\colon\mathbb{R}\times\Lambda_\sigma\to\mathbb{R}^b$$

defined by

$$\psi(\eta,\tau)_i=\langle g_i*G_\eta,h_\tau\rangle, \qquad \eta\neq 0,$$

$$\psi(0,\tau)_i=\langle g_i,h_\tau\rangle.$$

Then ψ is continuous by Theorem 2.3.3. Also (compare with θ in the proof of Lemma 2.4.1) ψ is a Frechet differentiable function of its second variable and its partial derivative $\psi_2'(\eta,\tau)\in\mathscr{L}(\mathbb{R}^\sigma,\mathbb{R}^b)$ is a continuous function of $(\eta,\tau)\in\mathbb{R}\times\Lambda_\sigma$ such that rank $\psi_2'(0,\bar{\tau})=b$. By the implicit function theorem there is a continuous mapping $u\colon(-\delta,\delta)\to\Lambda_\sigma$, defined on some interval $(-\delta,\delta)$, such that $u(0)=\bar{\tau}$ and $\psi(\eta,u(\eta))=0$ for $\eta\in(-\delta,\delta)$. Then

$$\begin{aligned}e_m(\mathscr{K}^{(\eta)},\infty)&\leqslant\|G_\eta*\bar{k}+K^{(\eta)}*h_{u(\eta)}\|_\infty\\&=\|G_\eta*(\bar{k}+(K*G_\eta)*h_{u(\eta)}\|_\infty\\&\leqslant\|\bar{k}+(K*G_\eta)*h_{u(\eta)}\|_\infty\\&\to\|\bar{k}+K*h_{\bar{\tau}}\|_\infty \qquad \text{as} \quad \eta\to 0,\\&=e_m(\mathscr{K},\infty).\end{aligned}$$

Now we have

$$e_m(\mathscr{K}, \infty) \leqslant \|P_0\|_\infty = \liminf e_m(\mathscr{K}^{(\eta)}, \infty) \leqslant \limsup e_m(\mathscr{K}^{(\eta)}, \infty)$$
$$\leqslant e_m(\mathscr{K}, \infty).$$

This proves that P_0 is extremal for $e_m(\mathscr{K}, \infty)$.

Finally, it follows from the fact that $\|P_{\eta_r} - G_{\eta_r} * P_0\| \to 0$ that if r is large the number of alternations of P_{η_r} on $[0, 1]$ is not greater than the number of alternations of P_0 on $[0, 1]$. To prove this it is necessary to examine the behaviour of the functions in neighbourhoods of 0 and 1; Theorem 2.3.3(iii) contains the information required. This proves (ii), and the proof of the theorem is complete.

3. Convolution Operators on Periodic Functions

3.1. *Statement of Result*

In this section $\tilde{C}$ will denote the space of continuous 2π-periodic real functions and $\tilde{L}^\infty$ the space of (equivalence classes of) bounded measurable functions. The $2m+1$ dimensional space of trigonometric polynomials of order $\leqslant m$ will be denoted by $\mathscr{E}_m$.

We shall be concerned with kernels K defined on $\mathbb{R} \times \mathbb{R}$, bounded, 2π-periodic in each variable separately and (for definiteness) piecewise continuous in each variable, such that there is an integral operator

$$T_K : \tilde{L}^\infty \to \tilde{C}$$

defined by

$$(T_K f)(s) = (K * f)(s) = \int_0^{2\pi} K(s, t) f(t)\, dt = \int_\alpha^{\alpha + 2\pi} K(s, t) f(t)\, dt.$$

If K is a kernel and $k_1, \dots, k_a$; $g_1, \dots, g_b$ are functions in $\tilde{C}$ then

$$\mathscr{K} = (K; k_1, \dots, k_a; g_1, \dots, g_b)$$

will be referred to as a periodic system. The notations of Section 2.1 are applicable.

The principal result of this section concerns convolution operators with kernels of the form $D(s - t)$, where D is a 2π-periodic function (there will be no confusion if we use D for both function and kernel) and particular systems of the form

$$\mathscr{D} = (D; p_1, \dots, p_a; p_1, \dots, p_a)$$

in which either $a=0$ and $\mathscr{D}=(D;\varnothing,\varnothing)$ or $p_1,\ldots,p_a$ are the functions 1, $\cos t$, $\sin t,\ldots,\cos mt$, $\sin mt$ for some non-negative integer $m=\frac{1}{2}(a-1)$. (In the notation of Section 1.2 we are concerned with the situations $M_a=N_b=\{0\}$ and $M_a=N_b=\mathscr{E}_m$). However the results are formulated only for the cases in which $a>0$. Only quite trivial changes are required to obtain corresponding statements and proofs for the case $a=0$.

Let

$$\tilde{\Lambda}_n=\{\tau=(\tau_1,\ldots,\tau_n)\in\mathbb{R}^n\colon \tau_1<\cdots<\tau_n<\tau_1+2\pi\}.$$

If n is an even integer and $\tau\in\tilde{\Lambda}_n$ let $\tilde{h}_\tau$ be the 2π-periodic step function defined by

$$\tilde{h}_\tau(t)=(-1)^i \qquad \text{for} \quad \tau_i<t<\tau_{i+1} \quad \text{and} \quad i=1,\ldots,n \;\; (\tau_{n+1}=\tau_1+2\pi).$$

The first condition we must formulate concerns only the function D (and the integer m) and does not correspond to any of the conditions of Section 2.1.

Condition ($\tilde{C}0$). If p is a trigonometric polynomial then $(D-p)^{-1}(0)$ contains no interval.

The remaining two conditions will be formulated for a periodic system $\mathscr{K}=(K;k_1,\ldots,k_a;g_1,\ldots,g_b)$ in which $b-a$ is an even integer.

Condition ($\tilde{C}1$). For each pair of non-negative integers σ and ρ, with $\sigma=\rho+\frac{1}{2}(a-b)$ and for either $\varepsilon_\sigma=1$ or $\varepsilon_\sigma=-1$

$$\varepsilon_\sigma\mathscr{K}\begin{pmatrix}1,\ldots,a;\ \xi_1,\ldots,\xi_{2\rho+1}\\ 1,\ldots,b;\ \tau_1,\ldots,\tau_{2\sigma+1}\end{pmatrix}\geqslant 0$$

for all $\xi=(\xi_1,\ldots,\xi_{2\rho+1})$ in $\tilde{\Lambda}_{2\rho+1}$ and $\tau=(\tau_1,\ldots,\tau_{2\sigma+1})$ in $\tilde{\Lambda}_{2\sigma+1}$. In the terminology of [6] this is a "sign regularity" condition.

Condition ($\tilde{C}2$). For each positive integer σ and each $\tau\in\tilde{\Lambda}_\sigma$ the functions $k_1,\ldots,k_a$, $K(\cdot,\tau_1),\ldots,K(\cdot,\tau_\sigma)$ are linearly independent.

In the first draft of this paper the systems $\mathscr{S}$ were required to satisfy a further condition, but by extending some arguments of [18] it can be shown to be a consequence of the other conditions. The implication will be stated here as a theorem and proved in Section 3.2.

3.1.1. THEOREM. *Let $\mathscr{S}=(D;p_1,\ldots,p_a;p_1,\ldots,p_a)$ be a periodic system in which $D\in\tilde{C}$, $a=2m+1$ and $\operatorname{sp}\{p_1,\ldots,p_a\}=\mathscr{E}_m$. If D satisfies ($\tilde{C}0$) and $\mathscr{S}$ satisfies ($\tilde{C}1$) and ($\tilde{C}2$) then for each integer $n>m$ there exists $\tau^0\in\tilde{\Lambda}_{2n}$ and $p_0\in\mathscr{E}_{n-1}$ such that $\tau^0_{j+1}-\tau^0_j=\pi/n$ for $j=1,\ldots,$ $2n-1$ and $\pm(D-p_0)\tilde{h}_{\tau^0}\geqslant 0$.*

The only explicit examples we know that satisfy the conditions we require are the following ones.

3.1.2. EXAMPLES. Let D_1 be the right continuous, 2π-periodic function defined by

$$D_1(t) = \frac{\pi - t}{2\pi} \qquad \text{for} \quad 0 \leqslant t < 2\pi.$$

For each integer $r \geqslant 2$ let

$$D_r(t) = \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k^r} \cos\left(kt - r\frac{\pi}{2}\right).$$

(This equation holds for $r = 1$ provided that $t/2\pi$ is not an integer).

The functions D_r are well known. D_r is piecewise polynomial with knots at $2k\pi$, k an integer. For each $r \geqslant 1$

$$D_{r+1}(t) = D_{r+1}(0) + \int_0^t D_r(u)\, du,$$

but more significantly from our point of view this relation takes the form

$$D_{r+1} = D_r * D_1.$$

Let $\mathscr{D}_r$ denote the periodic systems $\mathscr{D}_r = (D_r; 1, 1)$ (in which $a = 1$, $m = 0$). That D_r satisfies ($\tilde{C}0$) is obvious. Furthermore it is a standard result of approximation theory that D_r satisfies the conclusion of Theorem 3.1.1 (see, e.g., [10, Chap. 8]). It is easily verified that the system D_r satisfies condition ($\tilde{C}2$). It also satisfies ($\tilde{C}1$). More precisely we state as a

3.1.3. THEOREM. *For each positive integer r and non-negative integer* σ

$$-\mathscr{D}_r\begin{pmatrix} 1; & \xi_1, ..., \xi_{2\sigma+1} \\ 1; & \tau_1, ..., \tau_{2\sigma+1} \end{pmatrix} \geqslant 0$$

for all ξ *and* τ *in* $\tilde{\Lambda}_{2\sigma+1}$.

The theorem will be proved in Section 3.2.

We can now formulate the principal result of this section. It generalises those results of [21] which apply to the periodic situation.

3.1.4. THEOREM. *Let* $\mathscr{D} = (D; p_1, ..., p_a; p_1, ..., p_a)$ *be a periodic system in which* $D \in \tilde{C}$, $a = 2m + 1$ *and* $\text{sp}\{p_1, ..., p_a\} = \mathscr{C}_m$, *and let* T_D *denote the convolution operator in* $\mathscr{L}(\tilde{L}^\infty, \tilde{C})$.

Suppose that D satisfies ($\tilde{C}0$) *and that the system* $\mathscr{D}$ *satisfies* ($\tilde{C}1$) *and* ($\tilde{C}2$). *Then for each* $n > m$

$$d_{L^1}(D, \mathscr{E}_{n-1}) = a_{2n-1}(T_D; \mathscr{E}_m, \mathscr{E}_m^\perp) = a_{2n}(T_D; \mathscr{E}_m, \mathscr{E}_m^\perp)$$
$$= k_{2n-1}(T_D; \mathscr{E}_m, \mathscr{E}_m^\perp) = k_{2n}(T_D; \mathscr{E}_m, \mathscr{E}_m^\perp) = |(D * h_{\tau^0})(0)|,$$

where τ^0, *corresponding to n, is as in Theorem* 3.1.1.

There is $\xi^0 \in \tilde{A}_{2n}$ *such that* $\xi_{i+1}^0 - \xi_i^0 = \pi/n$ *for* $i = 1,\dots, 2n-1$ *and* $(D * \tilde{h}_{\tau^0})(\xi_i^0) = 0$ *for* $i = 1,\dots, n$. *Furthermore*

$$\mathscr{D}\begin{pmatrix} 1,\dots, a; & \xi_1^0,\dots, \xi_{2n}^0 \\ 1,\dots, a; & \tau_1^0,\dots, \tau_{2n}^0 \end{pmatrix} \neq 0.$$

Let p_0, *corresponding to n, be as in Theorem* 3.1.1. *Let* $\tilde{H}_0$ *be the kernel defined by*

$$\tilde{H}_0(s, t) = D(s-t) - \frac{\mathscr{D}\begin{pmatrix} 1,\dots, a; & \xi_1^0,\dots, \xi_{2n}^0, s \\ 1,\dots, a; & \tau_1^0,\dots, \tau_{2n}^0, t \end{pmatrix}}{\mathscr{D}\begin{pmatrix} 1,\dots, a; & \xi_1^0,\dots, \xi_{2n}^0 \\ 1,\dots, a; & \tau_1^0,\dots, \tau_{2n}^0 \end{pmatrix}}.$$

Let

$$\tilde{L}_0 = \left\{ q(s) + \sum_{j=1}^{2n} \beta_j D(s - \tau_j^0) : q \in \mathscr{E}_m, \sum_{j=1}^{2n} \beta_j p(\tau_j^0) = 0 \text{ for } p \in \mathscr{E}_m \right\}.$$

Then (i) p_0 *is a best approximation to D from* $\mathscr{E}_{n-1}$ *in the* L^1 *norm,*

(ii) $\mathscr{E}_{n-1}$ *is extremal for* $k_{2n-1}(T_D; \mathscr{E}_m, \mathscr{E}_m^\perp)$,

(iii) $\tilde{L}_0$ *is of dimension 2n and is extremal for* $k_{2n}(T_D; \mathscr{E}_m, \mathscr{E}_m^\perp)$,

(iv) $T_{\tilde{H}_0}$ *is extremal for* $a_{2n}(T_D; \mathscr{E}_m, \mathscr{E}_m^\perp)$.

3.1.4. *Remarks.* (1) T_{p_0} is not extremal for $a_{2n-1}(T_D; \mathscr{E}_m, \mathscr{E}_m^\perp)$ as $\mathscr{E}_m \not\subseteq p_0 * (\mathscr{E}_m^\perp) = T_{p_0}(\mathscr{E}_m^\perp)$ (cf. Lemma 3.3.1).

(2) The function D_1 is not continuous and so the system $\mathscr{D}_1 = (D_1; 1; 1)$ does not satisfy the conditions of the theorem as stated. It should be possible, but cumbersome, to formulate general conditions short of continuity, and appropriate variants of the results in Section 2.3 which would permit an extension of the theorem to include the system $\mathscr{D}_1$. However, the systems $\mathscr{D}_r$, $r \geqslant 2$, do satisfy the conditions stated.

(3) It is appropriate to comment on the particularity of the theorem. Suppose that one attempts to apply the arguments to a more general system $\mathscr{S} = (D; f_1,\dots, f_a;\ g_1,\dots, g_b)$. At one point one requires $a = b$. There is a crucial step in the (Tikhomirov's) argument which depends upon M_a and N_b being translation invariant. The only finite dimensional translation invariant

subspaces of $\tilde{C}$ are spaces of trigonometric polynomials. The final clauses (iii) and (iv) require that N_b satisfy a Chebyshev condition (corresponding to ($C3$) of Section 2.1). Therefore the combination of the arguments for all the conclusions of the theorem require that $N_b = \mathscr{F}_m$ and that M_a be a space of trigonometric polynomials. Little is sacrificed by considering only the situation $M_a = N_b = \mathscr{F}_m$.

(4) The sets $\mathscr{F} + D * \mathscr{F}_m^\perp$ are translation invariant. Therefore any translate of $\tilde{L}_0$ is extremal for $k_{2n}(T_D; \mathscr{F}_m, \mathscr{F}_m^\perp)$. Thus for the sets $\mathscr{F}_m + D * \mathscr{F}_m^\perp$ there are infinite families of subspaces extremal for some of the Kolmogorov widths.

Outline proof of Theorem 3.1.4. There is a long sequence of inequalities.

$$\begin{aligned}
|(D * \tilde{h}_{\tau^0})(0)| &\underset{(1)}{\geqslant} |D - p_0|_1 \underset{(2)}{\geqslant} \inf_{p \in \mathscr{F}_{n-1}} |D - p|_1 \underset{(3)}{=} \inf_{p \in \mathscr{F}_{n-1}} \|T_D - T_p\| \\
&\underset{(4)}{\geqslant} a_{2n-1}(T_D; \mathscr{F}_m, \mathscr{F}_m^\perp) \underset{(5)}{\geqslant} k_{2n-1}(T_D; \mathscr{F}_m, \mathscr{F}_m^\perp) \\
&\underset{(6)}{\geqslant} k_{2n}(T_D; \mathscr{F}_m, \mathscr{F}_m^\perp) \\
&\underset{(7)}{\geqslant} \|D * \tilde{h}_{\tau^0}\|_\infty \underset{(8)}{=} |D - \tilde{H}_0|_{\infty,\infty} \underset{(9)}{=} \|T_D - T_{\tilde{H}_0}\| \\
&\underset{(10)}{\geqslant} a_{2n}(T_D; \mathscr{F}_m, \mathscr{F}_m^\perp) \underset{(11)}{\geqslant} k_{2n}(T_D; \mathscr{F}_m, \mathscr{F}_m^\perp).
\end{aligned}$$

The conclusion of Theorem 3.1.1 is satisfied by D, m, $\tau^0 \in \tilde{\Lambda}^{2n}$ and $p_0 \in \mathscr{F}_{n-1}$. It follows that $\tilde{h}_{\tau^0}$ is orthogonal to $\mathscr{F}_{n-1}$. So equality (1) follows easily. Inequality (2) requires no comment. Equality (3) is by the case $q = \infty$ of Proposition 2.1.1. Inequality (4) is a consequence of Lemma 3.3.1. Inequalities (5) and (11) are cases of Lemma 1.2.1. Inequality (6) is immediate.

The major step in the proof of the theorem is the proof of inequality (7) which we will state and prove as Theorem 3.3.2. The argument is due to Tikhomirov [21]. Once we are in possssion of Theorem 3.2.1 the argument proceeds almost exactly as in the particular situation discussed by Tikhomirov.

At this point in the argument it follows that (1)–(7) are all equalities. Conclusion (i) of the theorem is immediate. It is simple to show that the subspace $\mathscr{F}_m + p_0 * \mathscr{F}_m^\perp \subseteq \mathscr{F}_{n-1}$ is extremal for $k_{2n-1}(T_D; \mathscr{F}_m, \mathscr{F}_m^\perp)$.

The step function $\tilde{h}_{\tau^0}$ has the property $\tilde{h}_{\tau^0}(t + \pi/n) = -\tilde{h}_{\tau^0}(t)$. So the continuous function $D * \tilde{h}_{\tau^0}$ has the same property and has zeros at the points of some $\xi^0 \in \tilde{\Lambda}^{2n}$ as described in the theorem. The assertion that

$$\mathscr{D}\begin{pmatrix} 1,\ldots, a; & \xi_1^0,\ldots, \xi_{2n}^0 \\ 1,\ldots, a; & \tau_1^0,\ldots, \tau_{2n}^0 \end{pmatrix} \neq 0$$

is a consequence of a periodic variant of Lemma 2.3.7. The kernel $\tilde{H}_0$ is now defined. Equality (8) (the right-hand side is the mixed norm of the kernel, as in Section 2.1) is a consequence of a periodic variant of Theorem 2.3.8. Equality (9) is by Proposition 2.1.1 again. The fact that $T_{\tilde{H}_0}$ satisfies the defining properties of $a_{2n}(T_D; \mathscr{C}_m, \mathscr{C}_m^{\perp})$ is a consequence of the variant of Theorem 2.3.8; this proves inequality (10). It now follows that there is equality throughout (1)–(11). The extremal property (iv) of $\tilde{H}_0$ is immediate; the properties of $\tilde{L}_0$ require another return to the variant of Theorem 2.3.8 and an appeal to Lemma 1.2.1(ii).

The periodic variants of the results in Section 2.3 are discussed in Section 3.2.

3.2. *Condition* ($\tilde{C}1$)

The entire Section 2.3 applies with little modification to the periodic situation. The modifications which are necessary will be described and the principal theorem, corresponding to Theorem 2.3.6, will be stated as Theorem 3.2.1. The latter part of this section is devoted to proofs of Theorems 3.1.1 and 3.1.3.

A large number of the changes which must be made in order to pass from the non-periodic to the periodic situation are simply the replacement of Λ by $\tilde{\Lambda}$. By this change we obtain conditions ($\tilde{C}3$), ($\tilde{C}4$), ($\tilde{C}5$) and ($\tilde{C}6$) corresponding to conditions ($C3$), ($C4$), ($C5$) and ($C6$) of Section 2.1. If ($\tilde{C}3$) is satisfied then b is either zero or an odd integer.

The one non-trivial change which has to be made is the replacement of the kernels G_n of 2.3.2 by the sequence of de la Vallee Poussin kernels

$$W_n(s, t) = \frac{1}{\binom{2n}{n}} \left(2 \cos \tfrac{1}{2}(s - t)\right)^{2n}.$$

These kernels have properties analogous to those of the kernels G_n, and in particular

$$W_n \begin{pmatrix} \xi_1, \ldots, \xi_{2\sigma+1} \\ \tau_1, \ldots, \tau_{2\sigma+1} \end{pmatrix} > 0$$

whenever $1 \leqslant \sigma \leqslant n$ and $\xi, \tau \in \tilde{\Lambda}_{2\sigma+1}$ (the result is due to Polya and Schoenberg [19], see also [6, Chap. 9, Section 3]). If $\mathscr{K}$ is a periodic system (with continuous kernel K) which satisfies ($\tilde{C}1$), ($\tilde{C}2$) and ($\tilde{C}3$) then for each integer σ it can be approximated by a system $W_n * \mathscr{K}$ satisfying a condition which we can describe as (Strict $\tilde{C}1(a + 2\sigma + 1)$). In this way we can obtain a periodic substitute for Theorem 2.3.3.

The number $S_c^-(f)$ of cyclic sign changes of a 2π-periodic function f is defined by

$$S_c^-(f) = \sup\{2n : f(\tau_i) f(\tau_{i+1}) < 0 \text{ for } i = 1,..., 2n \text{ and some } \tau \in \tilde{\Lambda}_{2n}\}.$$

The basic theorem of variation diminishing type is the

3.2.1. THEOREM. *Let $\mathscr{K}$ be a periodic system satisfying conditions ($\tilde{C}1$), ($\tilde{C}2$) and ($\tilde{C}3$). Suppose that $2\sigma + 1 \geqslant b$ and*

$$u = k_0 + K * f + \sum_{j=1}^{2\sigma} \kappa_j K(\cdot, \tau_j),$$

where $\tau \in \tilde{\Lambda}_{2\sigma}$, $k_0 \in M_a$, f is 2π-periodic and integrable over $[0, 2\pi]$, $f\tilde{h}_\tau \geqslant 0$, $f \in N_b^\perp$ and

$$\sum_{j=1}^{2\sigma} \kappa_j g(\tau_j) = 0 \qquad \text{for all} \quad g \in N_b.$$

(i) *If $f^{-1}(0)$ is a Lebesgue null set and u is zero at the points of $\xi \in \tilde{\Lambda}_n$ then $n \leqslant 2\rho = a - b + 2\sigma$. If u is zero at the points of $\xi \in \tilde{\Lambda}_{2\rho}$ then, for either $\varepsilon = 1$ or $\varepsilon = -1$, $\varepsilon u \tilde{h}_\xi \geqslant 0$.*

(ii) $S_c^-(u) \leqslant 2\rho = a - b + 2\sigma$.

Only trivial modifications to the statement of Lemma 2.3.4 and to the proofs of Lemma 2.3.4 and Theorem 2.3.6 are necessary. To obtain the periodic variants of Lemma 2.3.7 and Theorem 2.3.8 it is only necessary to make notational changes in the statements and proofs: substitute appropriate conditions, replace Λ by $\tilde{\Lambda}$, ρ and σ by 2ρ and 2σ, etc.

The rest of this section is devoted to the proofs of Theorems 3.1.1 and 3.1.3. The proof of Theorem 3.1.1 requires a sequence of lemmas. It will be supposed that $\mathscr{D} = (D; p_1,..., p_a ; p_1,..., p_a)$ is a periodic system in which $D \in \tilde{C}$, $a = 2m + 1$ and $\operatorname{sp}\{p_1,..., p_a\} = \mathscr{E}_m$. The conditions required for each lemma will be stated explicitly.

3.2.2. LEMMA. *Let $\sum \mu_n e^{int}$ be the complex Fourier series of the real function D. If $\mathscr{D}$ satisfies conditions ($\tilde{C}1$) and ($\tilde{C}2$) then $|\mu_n| \geqslant |\mu_{n+1}|$ for all $n \geqslant m + 1$. Consequently $\mu_n \neq 0$ for $n \geqslant m + 1$ and*

$$D * (\mathscr{E}_m^\perp \cap \mathscr{E}_n) = \mathscr{E}_m^\perp \cap \mathscr{E}_n$$

for $n \geqslant m + 1$.

The proof of the first assertion is a straightforward extension of the proof of [6, Chap. 5, Lemma 7.2]. It depends only upon Theorem 3.2.1 which is a consequence of ($\tilde{C}1$), ($\tilde{C}2$) (and ($\tilde{C}3$), which is also satisfied). If $\mu_n = 0$ for some $n \geqslant m+1$ then it follows that D is a trigonometric polynomial, which contradicts ($\tilde{C}2$). Thus $\mu_n \neq 0$ for $n \geqslant m+1$ and the final assertion follows easily.

3.2.3. LEMMA. *Suppose that $\mathscr{D}$ satisfies ($\tilde{C}1$) and ($\tilde{C}2$). If $n \geqslant m$ and $q \in \mathscr{F}_n$ then $S_c^-(D-q) \leqslant 2n+2$.*

The corresponding statement in the case $a=0$ is [18, Theorem 2.4]. The proof is an extension of that in [18]. Consider $q \in \mathscr{F}_n$. Then, by Lemma 3.2.2, $q = q_1 + D * q_2$ for some $q_1 \in \mathscr{F}_m$ and $q_2 \in \mathscr{F}_m^{\perp} \cap \mathscr{F}_n$. Define $f_r \in \tilde{L}^\infty$ by

$$f_r(t) = r, \qquad 0 \leqslant t < 1/r,$$
$$= 0, \qquad 1/r \leqslant t < 2\pi.$$

Then f_r can be written as $f_r = f_{r1} + f_{r2}$ with $f_{r1} \in \mathscr{F}_m$, $f_{r2} \in \mathscr{F}_m^{\perp}$. Then

$$D * f_r - q = (D * f_{r1} - q_1) + D * (f_{r2} - q_2)$$

and

$$D * f_{r1} - q_1 \in \mathscr{F}_m, \qquad f_{r2} - q_2 = f_r - (f_{r1} + q_2).$$

Now $f_{r1} + q_2 \in \mathscr{F}_n$ and $S_c^-(f_{r1} + q_2) \leqslant 2n$. Therefore, for all sufficiently large r, $S_c^-(f_{r2} - q_2) \leqslant 2n+2$ (one requires $r > \|f_{r1} + q_2\|$). It then follows from Theorem 3.2.1(ii) that $S_c^-(D * f_r - q) \leqslant 2n+2$. The conclusion of the lemma follows by going to the limit as $r \to \infty$.

If the function D satisfies ($\tilde{C}0$) then the conclusion of Lemma 3.2.3 can be strengthened.

3.2.4. LEMMA. *Suppose that D satisfies ($\tilde{C}0$) and that $\mathscr{D}$ satisfies ($\tilde{C}1$) and ($\tilde{C}2$). If $n \geqslant m+1$, $p_0 \in \mathscr{F}_{n-1}$ and $D - p_0$ is zero at the points of $\tau \in \tilde{A}_k$ then $k \leqslant 2n$. If $k = 2n$ then $\pm(D-p_0)\tilde{h} \geqslant 0$.*

Proof. It follows from condition ($\tilde{C}0$) that if $(D-p_0)(s) = 0$ then $D - p_0$ takes non-zero values in each of the intervals $(s-\delta, s)$ and $(s, s+\delta)$ for each $\delta > 0$.

Suppose that $(D-p_0)\tilde{h}_{\tau'} \geqslant 0$, where $\tau' \in \tilde{A}_{2\rho}$. By Lemma 3.2.3 there are such ρ and τ' and $2\rho \leqslant 2n$. If $2\rho = 2n$ then τ' must account for all the zeros of $D - p_0$, for otherwise $S_c^-(D - p_0 - \varepsilon) > 2n$ for some small ε of appropriate sign, and this contradicts Lemma 3.2.3. Thus if $2\rho = 2n$ then $k \leqslant 2n$. If $2\rho \leqslant 2n-2$ then there exists $p \in \mathscr{F}_{n-1}$ which is zero at the points of τ' and at no points other than translates of these by multiples of 2π. Suppose $[\tau_1', \tau_1' + 2\pi)$ contains r zeros of $D - p_0$ distinct from $\tau_1', \ldots, \tau_{2\rho}'$. Then for small ε of appropriate sign $S_c^-(D - p_0 + \varepsilon p) \geqslant 2\rho + 2r$. Therefore

$2\rho + r \leqslant 2\rho + 2r \leqslant 2n$. Thus $k \leqslant 2\rho + r \leqslant 2n$ and if $k = 2n$ then $r = 0$ and τ' can be replaced by τ. The proof of the lemma is complete.

3.2.5. LEMMA. *If $D \in \tilde{C}$ then for some s and some $p_0 \in \mathscr{E}_{n-1}$*

$$p_0\left(s + \frac{j}{n}\pi\right) = D\left(s + \frac{j}{n}\pi\right)$$

for $j = 0, 1, \ldots, 2n-1$.

Proof. Let

$$M = \left\{\left(p(0), p\left(\frac{1}{n}\pi\right), \ldots, p\left(\frac{2n-1}{n}\pi\right)\right) : p \in \mathscr{E}_{n-1}\right\} \subseteq \mathbb{R}^{2n}.$$

Then M is a hyperplane in $\mathbb{R}^{2n}$ and

$$M = \{(\xi_1, \ldots, \xi_{2n}) \in \mathbb{R}^{2n} : \alpha_1\xi_1 + \cdots + \alpha_{2n}\xi_{2n} = 0\}$$

for some $\alpha_1, \ldots, \alpha_{2n}$. Now $(1, \ldots, 1) \in M$ and therefore $\alpha_1 + \cdots + \alpha_{2n} = 0$. Define a mapping $\varphi\colon \mathbb{R} \to \mathbb{R}^{2n}$ by

$$\varphi(s) = \left(D(s), D\left(s + \frac{1}{n}\pi\right), \ldots, D\left(s + \frac{2n-1}{n}\pi\right)\right).$$

It will be shown that $\{\varphi(s)\colon s \in \mathbb{R}\}$ is not contained in either of the open half-spaces determined by M. For suppose on the contrary that

$$\alpha_1 D(s) + \alpha_2 D\left(s + \frac{1}{n}\pi\right) + \cdots + \alpha_{2n} D\left(s + \frac{2n-1}{n}\pi\right) > 0$$

for all $s \in \mathbb{R}$. Then, by taking

$$s = 0, \frac{1}{n}\pi, \ldots, \frac{2n-1}{n}\pi$$

in turn and summing, we obtain the inequality

$$(\alpha_1 + \cdots + \alpha_{2n})\left(D(0) + D)\left(\frac{1}{n}\pi\right) + \cdots + D\left(\frac{2n-1}{n}\pi\right)\right) > 0,$$

which contradicts the fact that $\alpha_1 + \cdots + \alpha_{2n} = 0$. That D is continuous implies that φ is continuous. Therefore, for some s, $\varphi(s) \in M$. That is, for some s and some $p \in \mathscr{E}_{n-1}$

$$D\left(s+\frac{j}{n}\pi\right)=p\left(\frac{j}{n}\pi\right) \qquad \text{for} \quad j=0,\dots,2n-1.$$

The conclusion of the lemma now follows with $p_0(t)=p(t-s)$.

Theorem 3.1.1 now follows from Lemmas 3.2.5 and 3.2.4.

We now proceed to the proof of Theorem 3.1.3. The first steps consist of computations for the system $\mathscr{D}_1=(D_1; 1; 1)$.

3.2.6. PROPOSITION. (i) *Let* $\tau_1<\tau_2<\cdots<\tau_\sigma<\tau_{\sigma+1}=\tau_1+2\pi$ *and* $\tau_1\leqslant u_1<u_2<\cdots<u_\sigma<\tau_1+2\pi$. *Then*

$$\mathscr{D}_1\begin{pmatrix}1;\ u_1,\dots,u_\sigma\\ 1;\ \tau_1,\dots,\tau_\sigma\end{pmatrix}=\begin{cases}-1 & \text{if } \tau_1\leqslant u_1<\tau_2\leqslant\cdots<\tau_\sigma\leqslant u_\sigma<\tau_1+2\pi,\\ 0 & \text{otherwise.}\end{cases}$$

(ii) *Let* $\tau_1<\tau_2<\cdots<\tau_{2\sigma+1}<\tau_{2\sigma+2}=\tau_1+2\pi$ *and* $\tau_1\leqslant u_1<u_2<\cdots<u_{2\sigma}<\tau_1+2\pi$. *If* $[\tau_j,\tau_{j+1})\cap\{u_1,\dots,u_\sigma\}=\varnothing$ *and* $[\tau_k,\tau_{k+1})\cap\{u_1,\dots,u_{2\sigma}\}$ *is one point for* $k\neq j$ *then*

$$\mathscr{D}_1\begin{pmatrix}1;\ u_1,\dots,u_{2\sigma}\\ \varnothing;\ \tau_1,\dots,\tau_{2\sigma+1}\end{pmatrix}=(-1)^{j+1}\frac{1}{2\pi}\int_{\tau_j}^{\tau_{j+1}}du.$$

(*The reason for the form of the right-hand side will be found in the proof of Theorem* 3.2.7.)

If $j\neq k$ *and* $([\tau_j,\tau_{j+1})\cup[\tau_k,\tau_{k+1}))\cap\{u_1,\dots,u_{2\sigma}\}=\varnothing$ *then*

$$\mathscr{D}_1\begin{pmatrix}1;\ u_1,\dots,u_{2\sigma}\\ \varnothing;\ \tau_1,\dots,\tau_{2\sigma+1}\end{pmatrix}=0.$$

Proof. (i) $[\tau_1,\tau_1+2\pi)=\bigcup_{j=1}^{\sigma}[\tau_j,\tau_{j+1})$. Either each subinterval contains one of the points $u_1,\dots,u_\sigma$ or there is a j such that $[\tau_j,\tau_{j+1})\cap\{u_1,\dots,u_\sigma\}=\varnothing$. In the latter case

$$D_1(u_i-\tau_j)-D_1(u_i-\tau_{j+1})=-(1/2\pi)(\tau_{j+1}-\tau_j), \qquad i=1,\dots,\sigma,$$

and in the $(\sigma+1)\times(\sigma+1)$ determinant

$$\mathscr{D}_1\begin{pmatrix}1;\ u_1,\dots,u_\sigma\\ 1;\ \tau_1,\dots,\tau_\sigma\end{pmatrix}$$

$\mathrm{Col}(j+2)-\mathrm{Col}(j+1)$ is a multiple of Col 1 and the determinant is zero (we interpret $\mathrm{Col}\,\sigma+2=\mathrm{Col}\ 1$). If $\tau_1\leqslant u_1<\tau_2\leqslant\cdots<\tau_\sigma\leqslant u_\sigma<\tau_1+2\pi$ then

$$D_1(u_i-\tau_j)-D_1(u_i-\tau_{j+1})=-\frac{1}{2\pi}(\tau_{j+1}-\tau_j) \qquad \text{for} \quad i\neq j.$$

$$=1-\frac{1}{2\pi}(\tau_{j+1}-\tau_j) \qquad \text{for} \quad i=j.$$

The conclusion is this case follows after performing in succession the column operations $\mathrm{Col}(j+1)+(1/2\pi)(\tau_{j+1}-\tau_j)\,\mathrm{Col}\,1$ for $j=2,3,\dots,\sigma-1$ and σ.

(ii) Suppose that $[\tau_j, \tau_{j+1}) \cap \{u_1,\dots,u_{2\sigma}\} = \varnothing$. Then

$$D_1(u_i-\tau_{j+1})-D_1(u_i-\tau_j)=\frac{\tau_{j+1}-\tau_j}{2\pi}$$

for $i=1,\dots,2\sigma$. If $k\neq j$ and also $[\tau_k,\tau_{k+1})\cap\{u_1,\dots,u_{2\sigma}\}=\varnothing$ then it is easily seen that

$$\mathscr{D}_1\begin{pmatrix}1;\ u_1,\dots,u_{2\sigma}\\ \varnothing;\ \tau_1,\dots,\tau_{2\sigma+1}\end{pmatrix}=0.$$

Now suppose that $[\tau_k,\tau_{k+1})\cap\{u_1,\dots,u_{2\sigma}\}$ is one point for $k\neq j$. Then, by the column operations $\mathrm{Col}(j+1)-\mathrm{Col}\,j$, we have

$$\mathscr{D}_1\begin{pmatrix}1;\ u_1,\dots,u_{2\sigma}\\ \varnothing;\ \tau_1,\dots,\tau_{2\sigma+1}\end{pmatrix}$$

$$=\begin{cases}(-1)^j\dfrac{\tau_{j+1}-\tau_j}{2\pi}\mathscr{D}_1\begin{pmatrix}1;\ u_1,\dots,u_{2\sigma}\\ 1;\ \tau_1,\dots,\hat{\tau}_{j+1},\dots,\tau_{2\sigma+1}\end{pmatrix} & \text{if } j=1,\dots,2\sigma\\[2ex] \dfrac{\tau_{j+1}-\tau_j}{2\pi}\mathscr{D}_1\begin{pmatrix}1;\ u_1,\dots,u_{2\sigma}\\ 1;\ \tau_2,\dots,\tau_{2\sigma+1}\end{pmatrix} & \text{if } j=2\sigma+1\end{cases}$$

$$=(-1)^{j+1}\frac{\tau_{j+1}-\tau_j}{2\pi},\qquad \text{for } j=1,\dots,2\sigma+1,$$

where the final equation is by (i) and the fact that

$$\mathscr{D}_1\begin{pmatrix}1;\ u_1,\dots,u_{2\sigma}\\ 1;\ \tau_2,\dots,\tau_{2\sigma+1}\end{pmatrix}=-\mathscr{D}_1\begin{pmatrix}1;\ u_2,\dots,u_{2\sigma},u_1+2\pi\\ 1;\ \tau_2,\dots,\tau_{2\sigma+1}\end{pmatrix}.$$

This completes the proof of (ii).

3.2.7. THEOREM. *If* $\mathscr{K}=(K;1;1)$ *let* $D\mathscr{K}$ *denote the system* $(K*D_1;1;1)$. *Then if* $\tau_1<\tau_2<\cdots<\tau_{2\sigma+1}<\tau_{2\sigma+2}=\tau_1+2\pi$,

$$(D\mathscr{K})\begin{pmatrix}1;\ \xi_1,\dots,\xi_{2\sigma+1}\\ 1;\ \tau_1,\dots,\tau_{2\sigma+1}\end{pmatrix}=\frac{1}{2\pi}\int_{\tau_1}^{\tau_2}\cdots\int_{\tau_{2\sigma+1}}^{\tau_{2\sigma+2}}\mathscr{K}\begin{pmatrix}1;\ \xi_1,\dots,\xi_{2\sigma+1}\\ 1;\ v_1,\dots,v_{2\sigma+1}\end{pmatrix}dv_1\cdots dv_{2\sigma+1}.$$

Proof. The determinant on the left can be expanded by its first row, and each term in the expansion by its first column. The basic composition formula (2.3.1) can then be applied to each term. The penultimate step depends upon Proposition 3.2.6(ii).

Let $\Delta = \{(u_1,\dots,u_{2\sigma}): \tau_1 \leqslant u_1 < u_2 < \cdots < u_{2\sigma} < \tau_1 + 2\pi\}$ and

$$\Delta_j = \{(u_1,\dots,u_{2\sigma}): \tau_1 \leqslant u_1 < u_2 < \cdots < u_{2\sigma} < \tau_1 + 2\pi,$$
$$\operatorname{card}[\tau_k, \tau_{k+1}) \cap \{u_1,\dots,u_{2\sigma}\} = 1 \text{ for } k \neq j\}.$$

Then

$$(D\mathscr{K})\begin{pmatrix}1; \xi_1,\dots,\xi_{2\sigma+1}\\ 1; \tau_1,\dots,\tau_{2\sigma+1}\end{pmatrix} = \sum_{i,j=1}^{2\sigma+1} (-1)^{i+j-1}(K * D_1)\begin{pmatrix}\xi_1,\dots,\hat{\xi}_i,\dots,\xi_{2\sigma+1}\\ \tau_1,\dots,\hat{\tau}_j,\dots,\tau_{2\sigma+1}\end{pmatrix}$$
$$= \sum_{i,j=1}^{2\sigma+1} (-1)^{i+j-1} \int\cdots\int_{\Delta} K\begin{pmatrix}\xi_1,\dots,\hat{\xi}_i,\dots,\xi_{2\sigma+1}\\ u_1,\dots,u_{2\sigma}\end{pmatrix}$$
$$\times D_1\begin{pmatrix}u_1,\dots,u_{2\sigma}\\ \tau_1,\dots,\hat{\tau}_j,\dots,\tau_{2\sigma+1}\end{pmatrix} du_1\cdots du_{2\sigma}$$
$$= -\int\cdots\int_{\Delta} \mathscr{K}\begin{pmatrix}\varnothing; \xi_1,\dots,\xi_{2\sigma+1}\\ 1; u_1,\dots,u_{2\sigma}\end{pmatrix} \mathscr{D}_1\begin{pmatrix}1; u_1,\dots,u_{2\sigma}\\ \varnothing; \tau_1,\dots,\tau_{2\sigma+1}\end{pmatrix} du_1\cdots du_{2\sigma}$$
$$= -\sum_{j=1}^{2\sigma+1} \int\cdots\int_{\Delta_j} \frac{(-1)^{j+1}}{2\pi}\left(\int_{\tau_j}^{\tau_{j+1}} du\right) \mathscr{K}\begin{pmatrix}\varnothing; \xi_1,\dots,\xi_{2\sigma+1}\\ 1; u_1,\dots,u_{2\sigma}\end{pmatrix} du_1\cdots du_{2\sigma}$$
$$= -\sum_{j=1}^{2\sigma+1} \int_{\tau_1}^{\tau_2}\cdots\int_{\tau_{2\sigma+1}}^{\tau_{2\sigma+2}} \frac{(-1)^{j+1}}{2\pi} \mathscr{K}\begin{pmatrix}\varnothing; \xi_1,\dots,\xi_{2\sigma+1}\\ 1; v_1,\dots,\hat{v}_j,\dots,v_{2\sigma+1}\end{pmatrix} dv_1\cdots dv_{2\sigma+1}.$$

The order of integration and summation can now be interchanged, and the conclusion of the theorem follows.

Proof of Theorem 3.1.3. First consider the case $r = 1$. Suppose that $\xi = (\xi_1,\dots,\xi_{2\sigma+1})$ and $\tau = (\tau_1,\dots,\tau_{2\sigma+1})$ are in $\tilde{\Lambda}_{2\sigma+1}$. If $\tau_1 \leqslant \xi_1$, $\xi_{2\sigma+1} < \tau_1 + 2\pi$ then, by Proposition 3.2.6(i)

$$-\mathscr{D}_1\begin{pmatrix}1; \xi_1,\dots,\xi_{2\sigma+1}\\ 1; \tau_1,\dots,\tau_{2\sigma+1}\end{pmatrix} \geqslant 0.$$

The general case follows by periodicity and cyclic permutation.

Now $D_{r+1} = D_r * D_1$ so the theorem follows by induction using Theorem 3.2.7.

3.3. *The Completion of the Proof of Theorem* 3.1.4

In this section inequalities (4) and (7) of Section 3.1 will be proved. Throughout this section it will be assumed that the conditions of Theorem 3.1.4 are satisfied. The integer $n > m$ will be fixed and $\tau^0 \in \tilde{\Lambda}_{2n}$ $(\tau^0_{j+1} - \tau^0_j = \pi/n$ for $j = 1,\dots, 2n-1)$ will be as in Theorem 3.1.1.

If $p \in \mathscr{E}_{n-1}$ then $p * \mathscr{E}_m^{\perp} \subseteq \mathscr{E}_{n-1} \cap \mathscr{E}_m^{\perp}$ and

$$\dim(p * \mathscr{E}_m^{\perp}) \leqslant (2n-1) - (2m-1).$$

Consequently the convolution operator T_p does not satisfy the defining conditions of

$$a_{2n-1}(T_D; \mathscr{E}_m, \mathscr{E}_m^{\perp}) = \inf\{\|T_D - T'\|: T' \in \mathscr{L}(\tilde{L}^{\infty}, \tilde{C}),$$
$$\dim T'(\mathscr{E}_m^{\perp}) \leqslant 2n-1, \mathscr{E}_m \subseteq T'(\mathscr{E}_m^{\perp})\}.$$

However inequality (4) of Section 3.1,

$$\inf_{p \in \mathscr{E}_{n-1}} \|T_D - T_p\| \geqslant a_{2n-1}(T_D; \mathscr{E}_m, \mathscr{E}_m^{\perp}),$$

follows from the following simple lemma which we state, without proof, in the notation of Section 1.2.

3.3.1. LEMMA. *Let $T' \in \mathscr{L}(E, F)$, $M_a \subseteq F$, $N_b^{\perp} \subseteq E$ and suppose that $\dim(T'(N_b^{\perp}) \cap M_a) = \alpha$ and $\dim T'(N_b^{\perp}) < \infty$. Then given $\varepsilon > 0$ there exists $T_\varepsilon \in \mathscr{L}(E, F)$ such that $\|T' - T_\varepsilon\| < \varepsilon$, $M_a \subseteq T_\varepsilon(N_b^{\perp})$ and*

$$\dim T_\varepsilon(N_b^{\perp}) \leqslant \dim T'(N_b^{\perp}) + (a - \alpha).$$

It now remains only to prove the

3.3.2. THEOREM. *If the conditions of Theorem 3.1.4 are satisfied then*

$$k_{2n}(T_D; \mathscr{E}_m, \mathscr{E}_m^{\perp}) \geqslant \|D * \tilde{h}_{\tau 0}\|_\infty.$$

If $D * \tilde{h}_{\tau 0} = 0$ there is nothing to prove, so we suppose $D * \tilde{h}_{\tau 0} \neq 0$.

The proof involves a sequence of lemmas. For each $s \in \mathbb{R}$ define $P_s: \tilde{C} \to \mathbb{R}^{2n}$ by

$$P_s f = \left(f(s), f\left(s + \frac{\pi}{n}\right), \ldots, f\left(s + \frac{2n-1}{n}\pi\right)\right).$$

The space $\mathbb{R}^{2n}$ will be given the max norm.

3.3.3. LEMMA. *If L is a subspace of $\tilde{C}$ and $\dim L = 2n$ then, for some $s \in \mathbb{R}$, $P_s(L) \neq \mathbb{R}^{2n}$.*

Proof. Let $f_1, \ldots, f_{2n}$ be a basis of L and define

$$\Delta(s) = \det\left(f_i\left(s + \frac{j-1}{n}\pi\right)\right)_{1 \leqslant i, j \leqslant 2n}.$$

Then Δ is continuous, $\Delta(s+\pi/n) = -\Delta(s)$ and therefore $\Delta(s) = 0$ for some $s \in \mathbb{R}$. The conclusion of the lemma now follows.

3.3.4. LEMMA. *Let F be the linear space of functions $f \in \tilde{L}^{\infty} \cap \mathscr{E}_m^{\perp}$ such that f is a step function with points of discontinuity contained in those of $\tilde{h}_{\tau^0}$. Then* $\dim F = 2n - (2m+1)$.

Proof. Let χ_j be the 2π-periodic function defined by

$$\begin{aligned} \chi_j(t) &= 1 \qquad \text{for} \quad t \in [\tau_j^0, \tau_j^0 + \pi/n) \\ &= 0 \qquad \text{for} \quad t \in [\tau_k^0, \tau_k^0 + \pi/n),\ k \neq j. \end{aligned}$$

Then $F = \mathscr{E}_m^{\perp} \cap \operatorname{sp}\{\chi_1,..., \chi_{2n}\}$. A direct calculation shows that the matrix $(\langle \chi_j, p_i \rangle)_{1 \leqslant i \leqslant a, 1 \leqslant j \leqslant 2n}$ (recall that $\mathscr{E}_m = \operatorname{sp}\{p_1,..., p_a\}$) is of rank $a = 2m+1$. The conclusion of the lemma follows.

3.3.5. LEMMA. *Let s be a point such that $|(D * \tilde{h}_{\tau^0})(s)| = \|D * \tilde{h}_{\tau^0}\|_{\infty}$. If $f \in F$ and $k \in \mathscr{E}_m$ then*

$$\|P_s(k + D * f)\|_{\infty} \geqslant \|D * \tilde{h}_{\tau^0}\|_{\infty} \|f\|_{\infty}.$$

Proof. Note that $(D * \tilde{h}_{\tau^0})(s + j/n) = \pm(-1)^j \|D * \tilde{h}_{\tau^0}\|_{\infty}$. Suppose that there exists $f \in F$ with $\|f\|_{\infty} = 1$ and $k \in \mathscr{E}_m$ such that

$$\|P_s(k + D * f)\|_{\infty} < \|D * \tilde{h}_{\tau^0}\|_{\infty}.$$

Then, by considering the values of the functions $D * \tilde{h}_{\tau^0} \pm (k + D * f)$ at $s + (j/n)\pi$ $(j = 0,..., 2n-1)$, we find that

$$S_c^-(D * \tilde{h}_{\tau^0} \pm (k + D * f)) \geqslant 2n$$

for either choice of sign. Now $f \in F$ and $\|f\|_{\infty} = 1$ so f must take one of the values ± 1 on one of the intervals of constancy $(\tau_j^0, \tau_j^0 + \pi/n)$ $(j = 1,..., 2n)$. So we choose $\varepsilon = \pm 1$ so that $\tilde{h}_{\tau^0} + \varepsilon f$ is zero on one of these intervals. It now follows from Theorem 3.2.1(ii) that

$$S_c^-(k + D * (\tilde{h}_{\tau^0} + \varepsilon f)) \leqslant 2n - 2.$$

This is a contradiction and the lemma is proved.

3.3.6. LEMMA. *Let $W = \mathscr{E}_m + D * (\mathscr{E}_m^{\perp} \cap (L^{\infty})_1)$ and let s be as in Lemma* 3.3.5. *Then*

$$P_s(W) \supseteq \|D * \tilde{h}_{\tau^0}\|_{\infty} \cdot (\mathbb{R}^{2n})_1.$$

Proof. Consider the composite linear mapping

$$\mathscr{E}_m \times F \xrightarrow{(k,f)\to k+D*f} \tilde{C} \xrightarrow{P_s} \mathbb{R}^{2n}.$$

The composite is injective by Lemma 3.3.5 and the Chebyshev (Haar) property of $\mathscr{E}_m$. By Lemma 3.3.4, $\dim \mathscr{E}_m \times F = 2n$ and so the composite mapping is an isomorphism.

If $z \in \mathbb{R}^{2n}$ and $\|z\|_\infty \leqslant \|D * \tilde{h}_{\tau 0}\|_\infty$ then $z = P_s(k + D * f)$ for some $(k, f) \in \mathscr{E}_m \times F$ and, by Lemma 3.3.5,

$$\|D * \tilde{h}_{\tau 0}\|_\infty \geqslant \|z\|_\infty \geqslant \|D * \tilde{h}_{\tau 0}\|_\infty \|f\|_\infty.$$

Therefore $\|f\|_\infty \leqslant 1$ and $k + D * f \in W$.

Completion of the Proof of Theorem 3.3.2

If $T_\lambda : \tilde{C} \to \tilde{C}$ is the translation operator defined by $(T_\lambda f)(s) = f(s - \lambda)$ then T_λ is an isometric isomorphism and $T_\lambda(W) = W$.

Suppose that L is a subspace of $\tilde{C}$ and $\dim L = 2n$. It must be shown that

$$\delta(W, L) \geqslant \|D * \tilde{h}_{\tau 0}\|_\infty.$$

Now $\delta(W, L) = \delta(W, T_\lambda L)$. So, by Lemma 3.3.3, we can replace L by $T_\lambda(L)$, for a suitable choice of λ, and we may suppose that $P_s(L) \neq \mathbb{R}^{2n}$. Now, using Lemma 3.3.6,

$$\begin{aligned}\delta(W, L) &\geqslant \delta(P_s(W), P_s(L))\\ &\geqslant \delta(\|D * \tilde{h}_{\tau 0}\| (\mathbb{R}^{2n})_1, P_s(L))\\ &= \|D * \tilde{h}_{\tau 0}\| \cdot \delta((\mathbb{R}^{2n})_1, P_s(L))\\ &= \|D * \tilde{h}_{\tau 0}\|.\end{aligned}$$

The proof is complete.

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